# Parallel Algorithms for Graphs on a Very Large Number of Nodes

**Krzysztof Onak**

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# Outline

1. Model of Computation
2. Sample Algorithms and Their Limitations
3. Efficiently Estimating MST Weight
4. Computing MST in Geometric Setting

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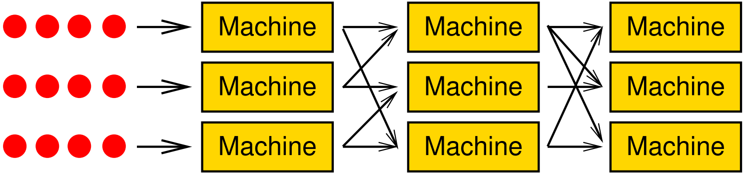

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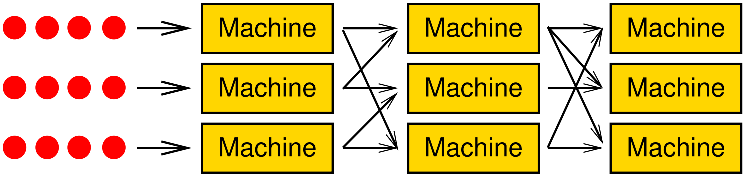

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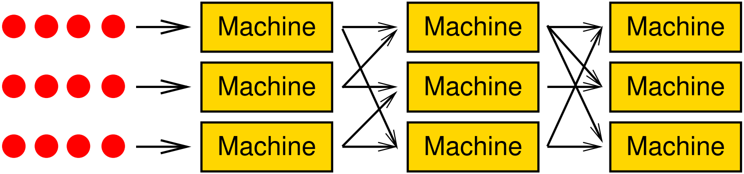


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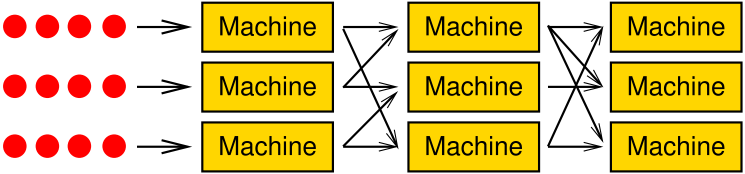


- Initially: each machine receives $n/m$ items
- Single round:
  1. Each machine performs computation
  2. Each machine sends and receives at most $O(s)$ data

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Goals:

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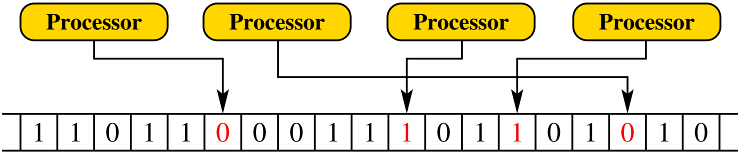
- Use amount of memory as close to linear as possible

# Comparison to PRAM

- PRAM: classic parallel model
  - *m* processors
  - processors access common memory



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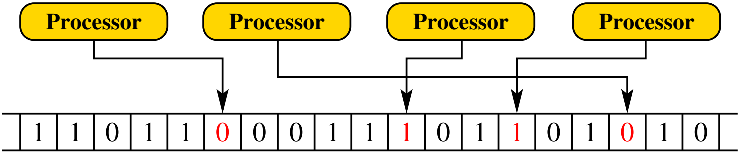

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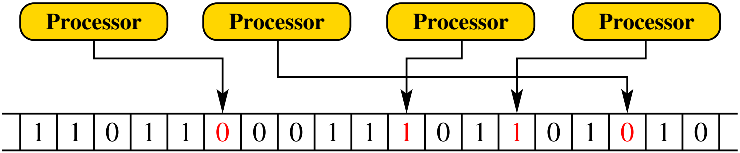

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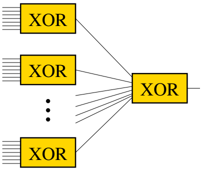

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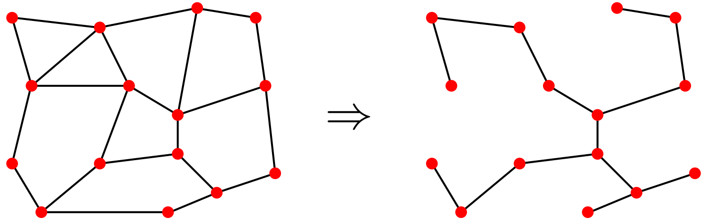
- Our goal: constant number of communication rounds

# Outline

# Main Subject of Study: Minimum Spanning Tree



Select the subset of edges of minimum weight

that connects all vertices

# Filtering Technique

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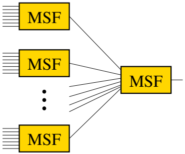

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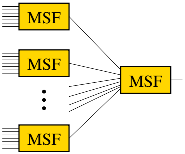


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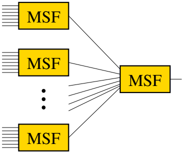


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- Complexity: $s = N^{1+\Omega(1)} \quad \Rightarrow \quad O(1)$ rounds

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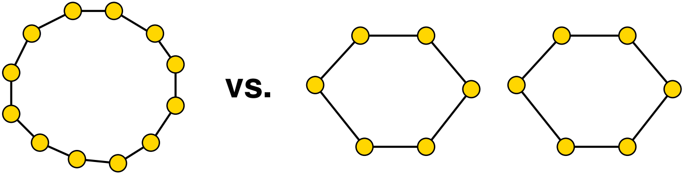

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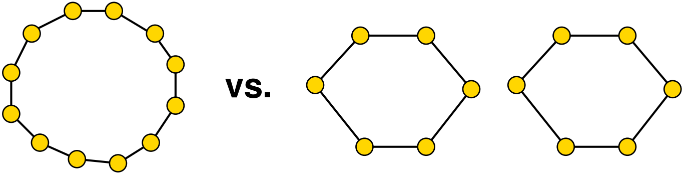

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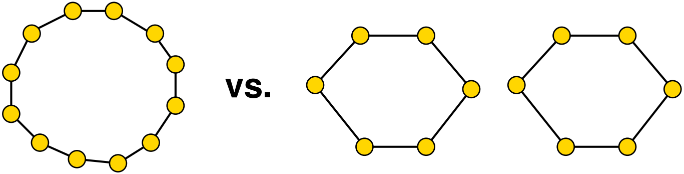


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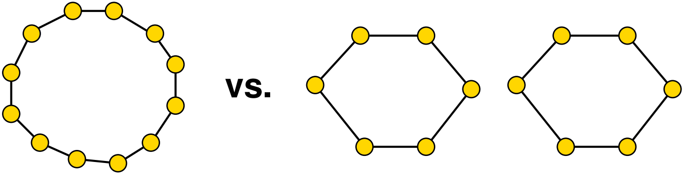


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- This talk: algorithms with $O(N^{\epsilon})$ space per machine

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- Input: $M$ edges, weights in $\{1, 2, \ldots, W\}$
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- Note: No dependence on $W$ would disprove
  Sparse Connectivity Conjecture

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- Good approximation:
  - Compute sizes of small components
  - Replace $1/C_i(v)$ with 0 if $C_i(v) \geq W/\epsilon$

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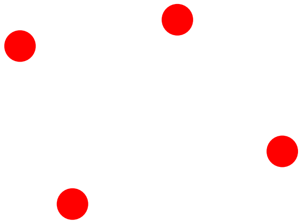
- $O(\log(W/\epsilon))$ updates suffice to explore useful nodes up to distance $W/\epsilon$
- Use QuickSort-like sorting algorithm of Goodrich, Sitchinava, Zhang (2011) to organize communication

# Outline

# Geometric Setting

Input: set of points in low dimensional metric space



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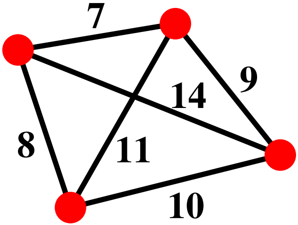


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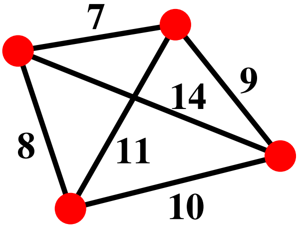


- Points induce a weighted graph
- Graph problems to consider:
  - Minimum Spanning Tree
  - Earth Mover Distance
  - Transportation Problem
  - Travelling Salesman Problem
  - ...

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[Andoni, Nikolov, O., Yaroslavtsev 2014]

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  - Running time: near-linear

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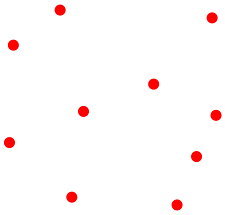
We reuse the Arora-Mitchell approach:

Apply a randomly shifted grid



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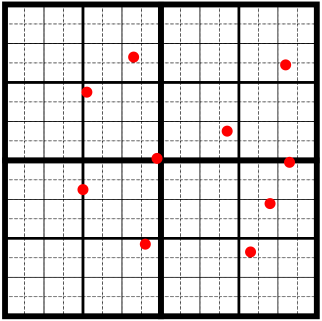

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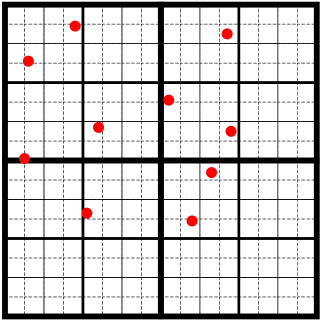

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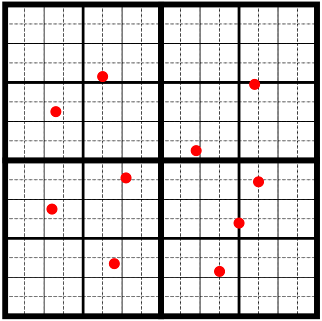

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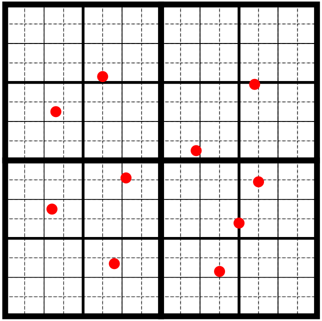

Key property: cell of side $\Delta$ separates points $x$ and $y$ w.p. $O(1) \cdot \frac{\rho(x,y)}{\Delta}$

# Using Random Gridding

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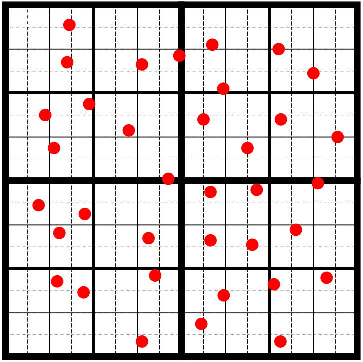

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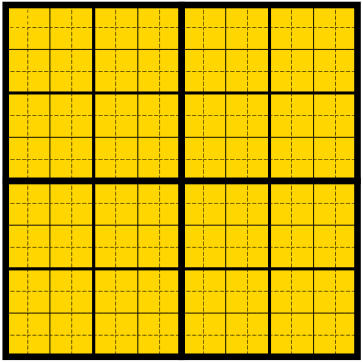

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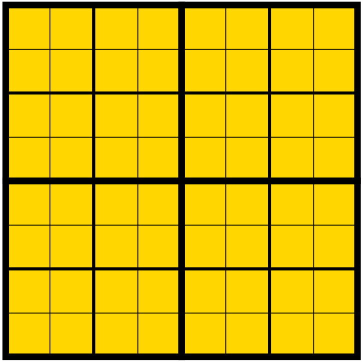

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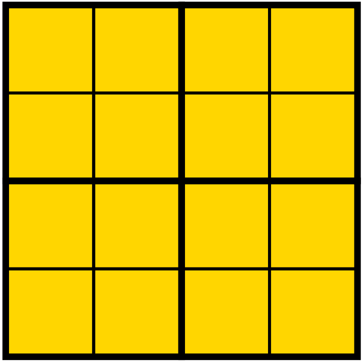

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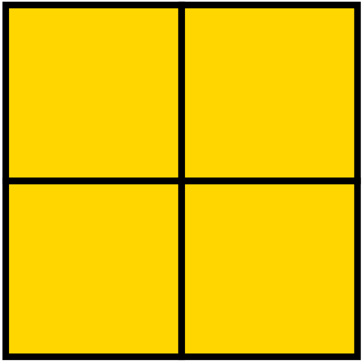

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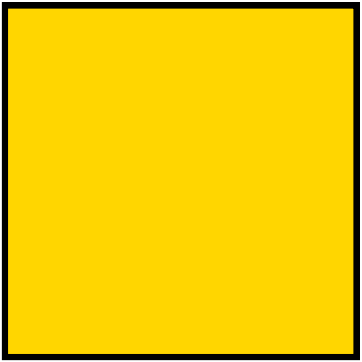

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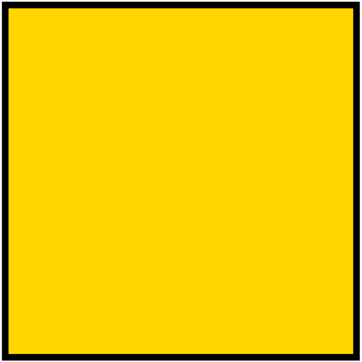

Can partially isolate what happens inside a cell

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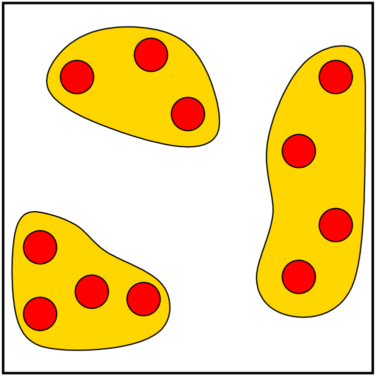

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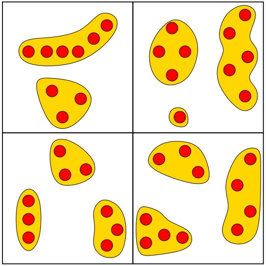

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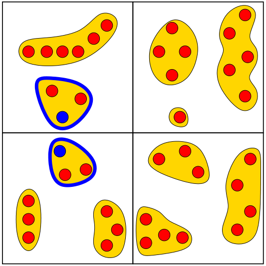

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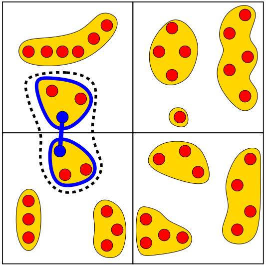

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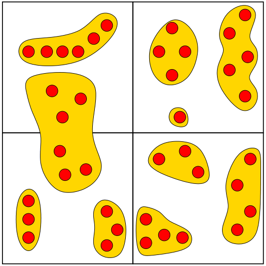

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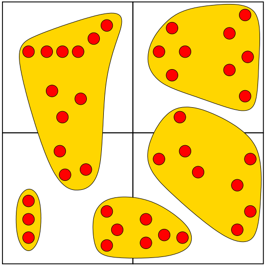

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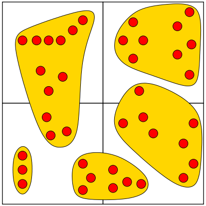

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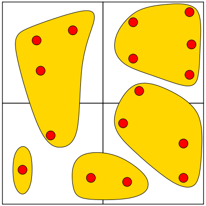

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- Expected cost of solution: optimum $\cdot$ $(1 + \epsilon \cdot \#\text{levels})$

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- Near-linear time:
  - Relax Kruskal's algorithm
  - Efficient nearest neighbor data structure [Krauthgamer, Lee 2004], [Cole, Gottlieb 2006]

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MST weight:

- Connected: $\leq 2(M - 1)$
- Not connected: $\geq 2M$

# Other Results

[Andoni, Nikolov, O., Yaroslavtsev 2014]

- Algorithm for approximating Earth-Mover Distance
- A new way of partitioning the instance into subproblems
- Resolves an open question of Sharathkumar & Agarwal (2012) about the transportation problem:

  First near-linear time algorithm

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- Future research:
  - More such algorithms
  - Better understanding of our limitations

# Questions?