Parallel Algorithms for Geometric Graph Problems

Grigory Yaroslavtsev

361 Levine

http://grigory.us

STOC 2014, joint work with Alexandr Andoni, Krzysztof Onak and Aleksandar Nikolov.

Theory Seminar: Fall 14

- Fridays 12pm—1pm
- 512 Levine / 612 Levine
  - Homepage: http://theory.cis.upenn.edu/seminar/
  - Google Calendar: see link above
- Announcements:
  - Theory list: http://lists.seas.upenn.edu/mailman/listinfo/theory-group

“The Big ~~Bang~~ Data Theory”

What should TCS say about big data?

- Usually:
  - **Running time**: (almost) linear, sublinear, …
  - **Space**: linear, sublinear, …
  - **Approximation**: $(1 + \epsilon)$, best possible, …
  - **Randomness**: as little as possible, …

- Special focus today: **round** complexity

Round Complexity

Information-theoretic measure of performance

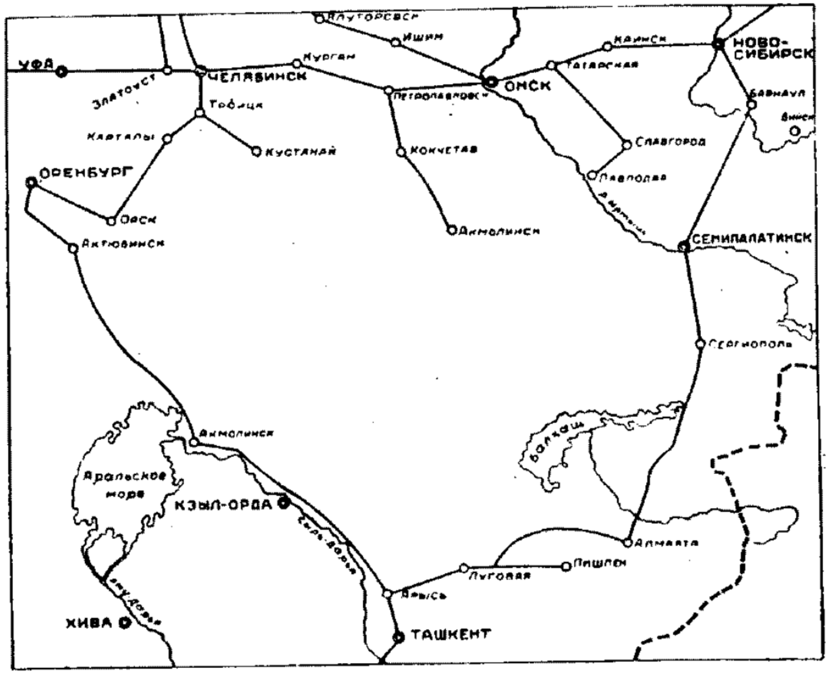
- Tools from information theory (Shannon'48)
- Unconditional results (lower bounds)

Example today:

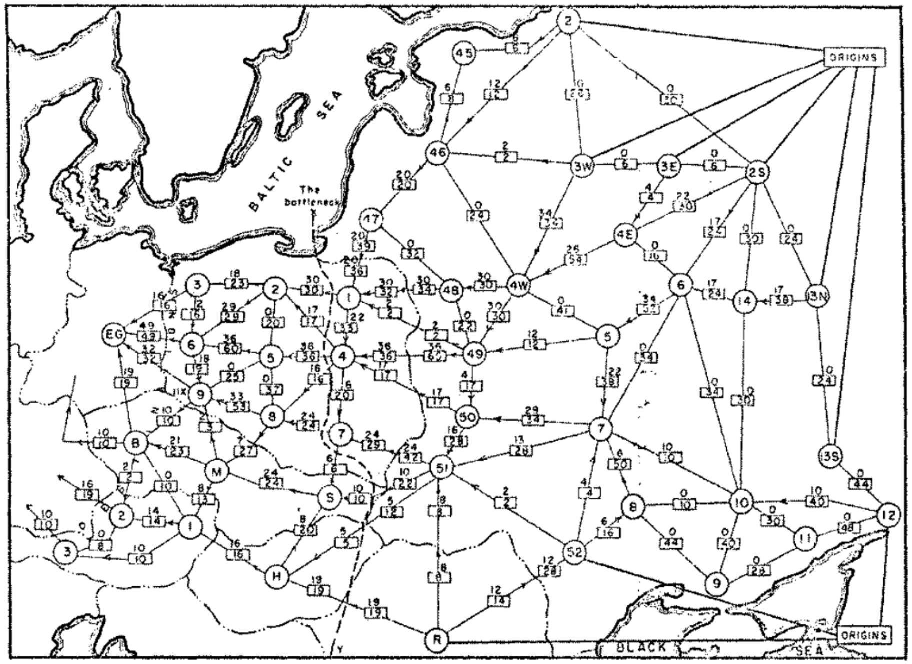
- Approximating Geometric Graph Problems

Approximation in Graphs

1930-50s: Given a graph and an optimization problem…



Transportation Problem:
Tolstoi [1930]



Minimum Cut (RAND):
Harris and Ross [1955] (declassified, 1999)

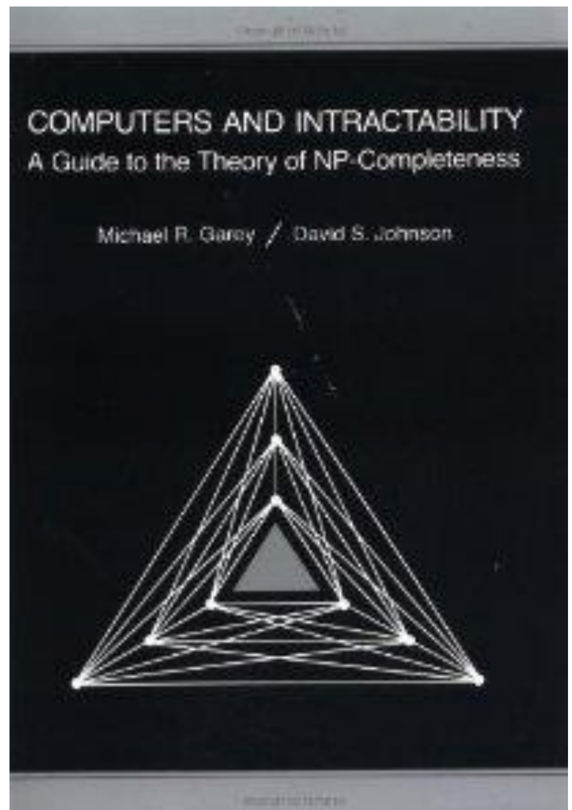
Approximation in Graphs

1960s: Single processor, main memory (IBM 360)

Approximation in Graphs

1970s: NP-complete problem – hard to solve exactly in time polynomial in the input size

"Black Book"



Approximation in Graphs

Approximate with multiplicative error $\boldsymbol{\alpha}$ on the **worst-case** graph G:

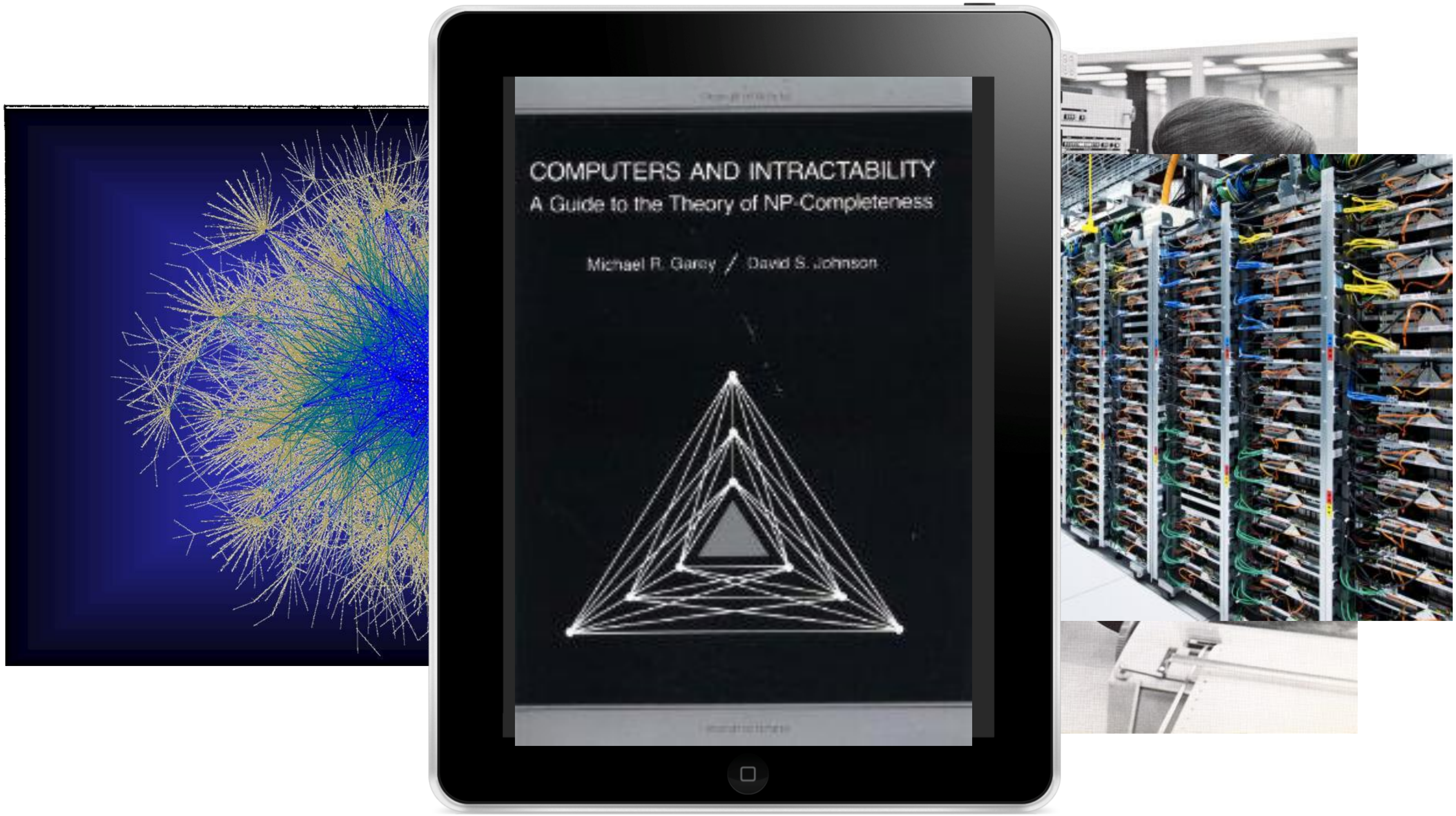
$$max_G \frac{Algorithm(G)}{Optimum(G)} \leq \boldsymbol{\alpha}$$

Generic methods:

- Linear programming
- Semidefinite programming
- Hierarchies of linear and semidefinite programs
- Sum-of-squares hierarchies
- …

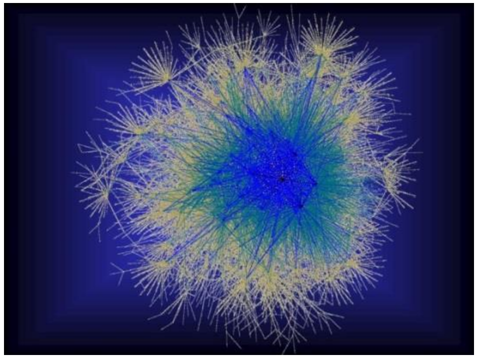
The New: Approximating Geometric Problems in Parallel Models

1930-70s to 2014



The New: Approximating Geometric Problems in Parallel Models

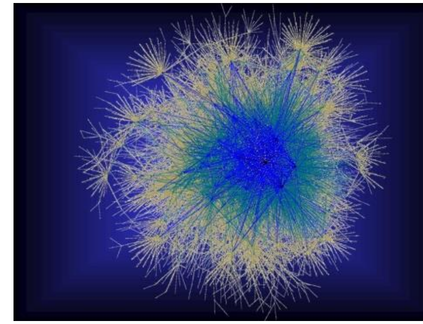
Geometric graph (implicit):
Euclidean distances between **n** points in $\mathbb{R}^{d}$



Already have solutions for old NP-hard problems (Traveling Salesman, Steiner Tree, etc.)

- Minimum Spanning Tree (clustering, vision)
- Minimum Cost Bichromatic Matching (vision)

Geometric Graph Problems

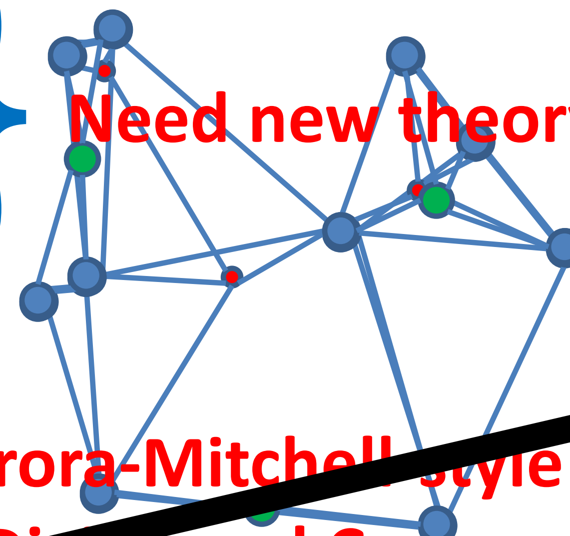


Combinatorial problems on graphs in $\mathbb{R}^d$

Polynomial time ("easy")

- Minimum Spanning Tree
- Earth-Mover Distance =

Min Weight Bi-chromatic Matching



Need new theory!

NP-hard ("hard")

- Steiner Tree
- Traveling Salesman
- Clustering (k-medians, facility location, etc.)

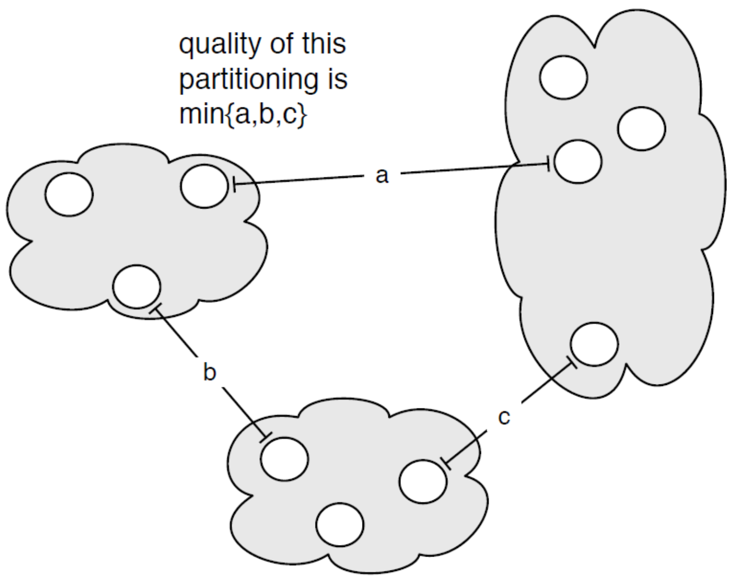
Arora-Mitchell-style "Divide and Conquer", easy to implement in Massively Parallel Computational Models

MST: Single Linkage Clustering

- [Zahn'71] **Clustering** via MST (Single-linkage):

k clusters: remove $\boldsymbol{k-1}$ longest edges from MST

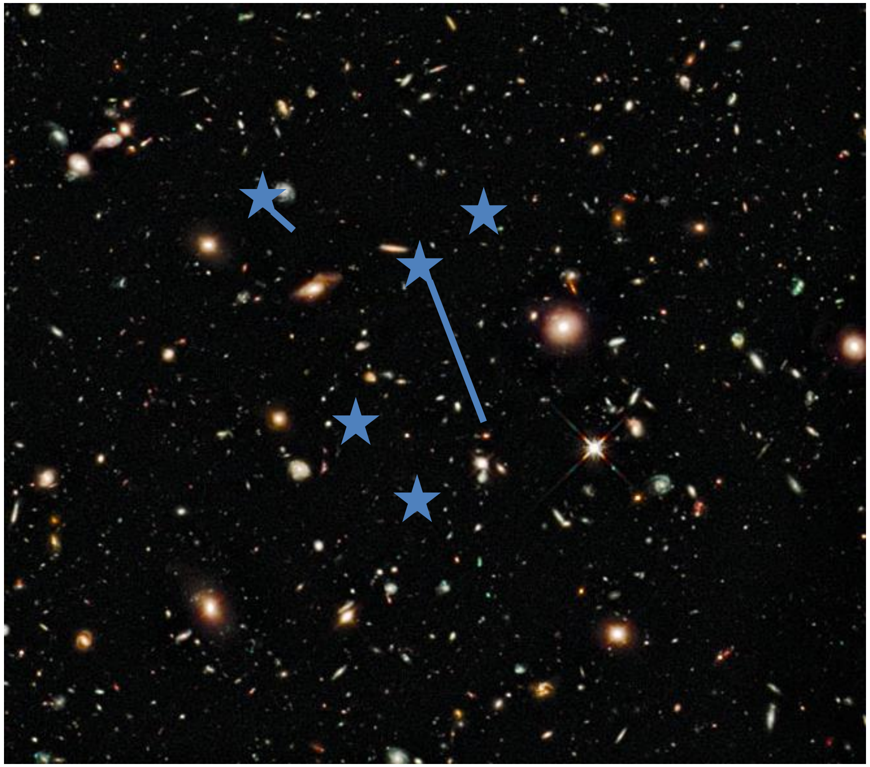
- Maximizes **minimum** intercluster distance



[Kleinberg, Tardos]

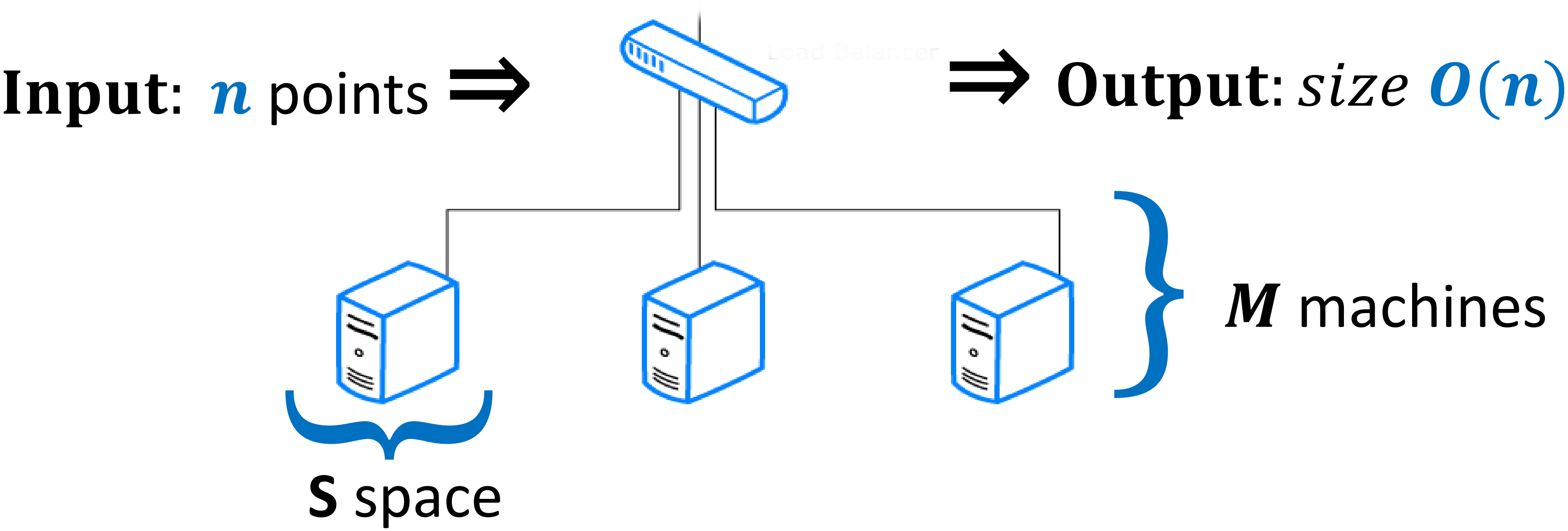
Earth-Mover Distance

- Computer vision: compare two pictures of moving objects (stars, MRI scans)



Computational Model

- **Input**: n points in a d-dimensional space (d constant)
- M machines, space S on each ($S = n^{\alpha}$, $0 < \alpha < 1$)
  - Constant overhead in total space: $M \cdot S = O(n)$
- **Output**: solution to a geometric problem (size O(n))
  - Doesn't fit on a single machine ($S \ll n$)

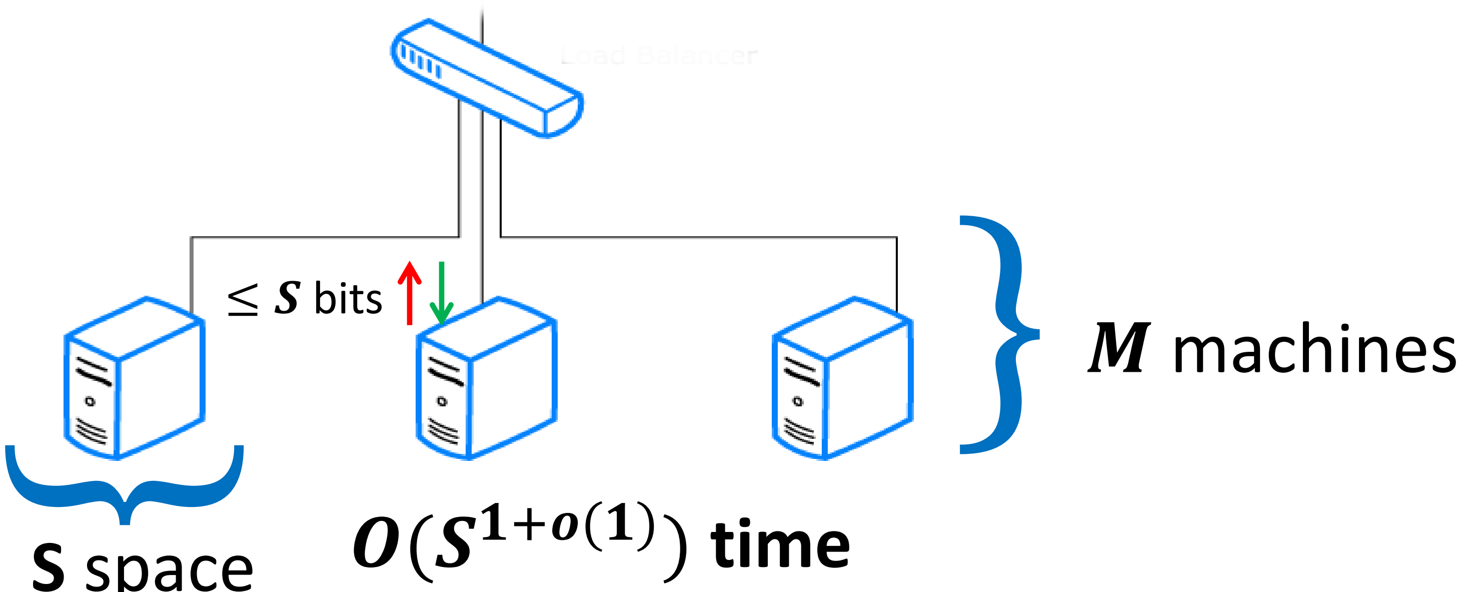


Computational Model

- Computation/Communication in $\boldsymbol{R}$ rounds:
  - Every machine performs a **near-linear time** computation => Total running time $O(\boldsymbol{n}^{\mathbf{1}+\boldsymbol{o}(\mathbf{1})}\boldsymbol{R})$
  - Every machine **sends/receives at most $\boldsymbol{S}$ bits** of information => Total communication $O(\boldsymbol{nR})$.

Goal: Minimize $\boldsymbol{R}$.

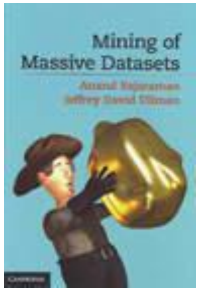
Our work: $\boldsymbol{R}$ = constant.



MapReduce-style computations

What I won't discuss today

- PRAMs (**shared memory**, multiple processors) (see e.g. [Karloff, Suri, Vassilvitskii'10])
  - Computing XOR requires $\widetilde{\Omega}(\log n)$ rounds in CRCW PRAM
  - Can be done in $O(\log_s n)$ rounds of MapReduce
- Pregel-style systems, Distributed Hash Tables (see e.g. Ashish Goel's class notes and papers)
- Lower-level implementation details (see e.g. Rajaraman-Leskovec-Ullman book)



Models of parallel computation

- **Bulk-Synchronous Parallel Model** (BSP) **[Valiant,90]**

  Pro: Most general, generalizes all other models

  Con: Many parameters, hard to design algorithms

- **Massive Parallel Computation** [Feldman-Muthukrishnan-Sidiropoulos-Stein-Svitkina’07, Karloff-Suri-Vassilvitskii’10, **Goodrich-Sitchinava-Zhang’11, ..., Beame, Koutris, Suciu’13**]

  Pros:

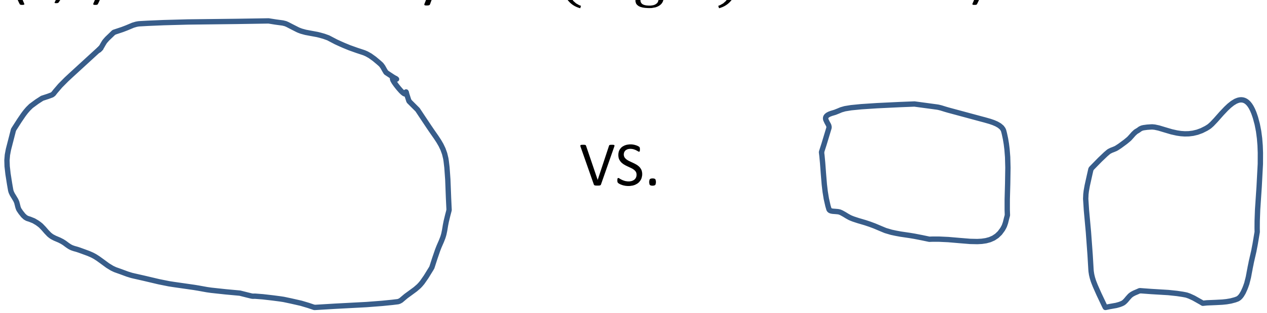
  - Inspired by **modern** systems (Hadoop, MapReduce, Dryad, …)
  - Few parameters, **simple** to design algorithms
  - **New algorithmic ideas**, robust to the exact model specification
  - **# Rounds** is an information-theoretic measure => can prove unconditional lower bounds
  - Between **linear sketching** and **streaming with sorting**

Previous work

- **Dense graphs** vs. sparse graphs
  - **Dense**: $\boldsymbol{S} \gg \boldsymbol{n}$ (or $\boldsymbol{S} \gg$ solution size)

    "Filtering" (Output fits on a single machine) [Karloff, Suri Vassilvitskii, SODA'10; Ene, Im, Moseley, KDD'11; Lattanzi, Moseley, Suri, Vassilvitskii, SPAA'11; Suri, Vassilvitskii, WWW'11]
  - Sparse: $\boldsymbol{S} \ll \boldsymbol{n}$ (or $\boldsymbol{S} \ll$ solution size)

    Sparse graph problems appear hard (**Big open question**: (s,t)-connectivity in $o(\log n)$ rounds?)



Large geometric graphs

- Graph algorithms: **Dense graphs** vs. sparse graphs
  - **Dense**: $\boldsymbol{S} \gg \boldsymbol{n}$.
  - Sparse: $\boldsymbol{S} \ll \boldsymbol{n}$.

- **Our setting:**
  - Dense graphs, sparsely represented: $O(\mathbf{n})$ space
  - Output doesn't fit on one machine ($\boldsymbol{S} \ll \boldsymbol{n}$)
- **Today:** $(1+\epsilon)$-approximate MST
  - $\boldsymbol{d} = 2$ (easy to generalize)
  - $\boldsymbol{R} = \log_{\boldsymbol{S}} \boldsymbol{n} = O(1)$ rounds ($\boldsymbol{S} = \boldsymbol{n}^{\boldsymbol{\Omega(1)}}$)

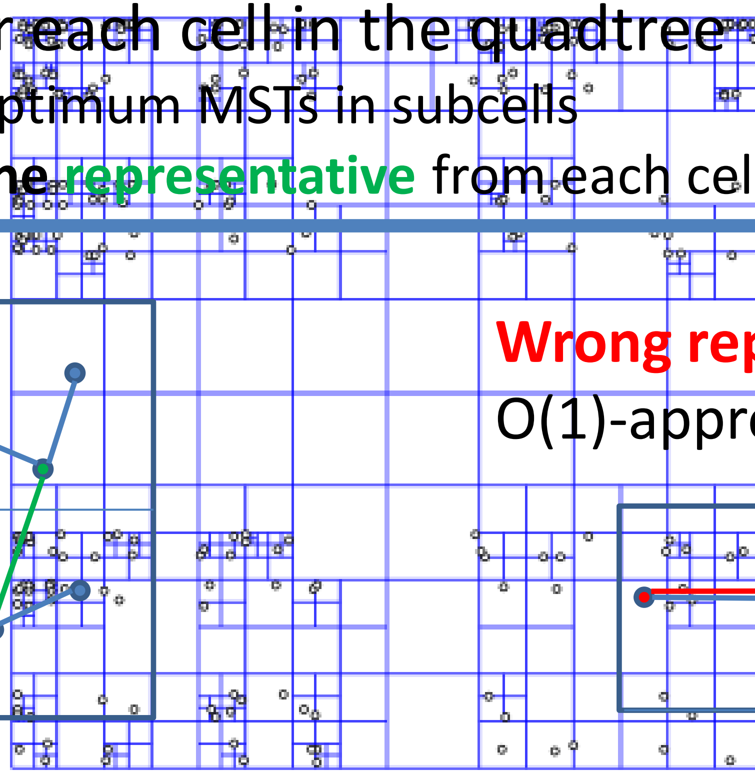
$O(\log n)$-MST in $R = O(\log n)$ rounds

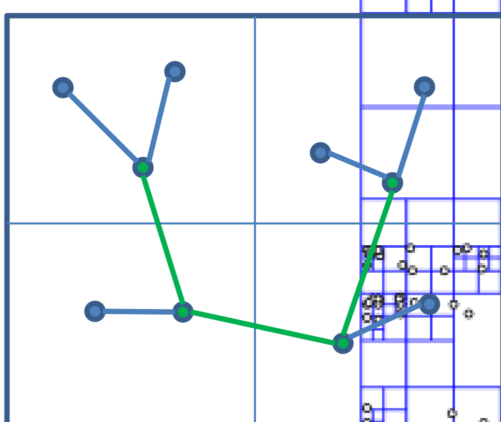
- Assume points have integer coordinates $[0, \dots, \Delta]$, where $\Delta = O(n^2)$.

Impose an $O(\log n)$-depth quadtree

Bottom-up: For each cell in the quadtree

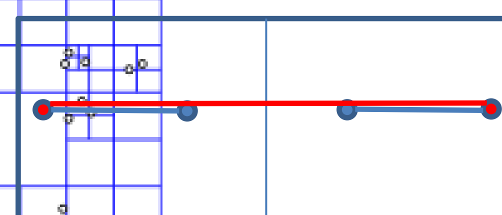
- compute optimum MSTs in subcells
- Use only **one** representative from each cell on the next level





Wrong representative:

O(1)-approximation per level



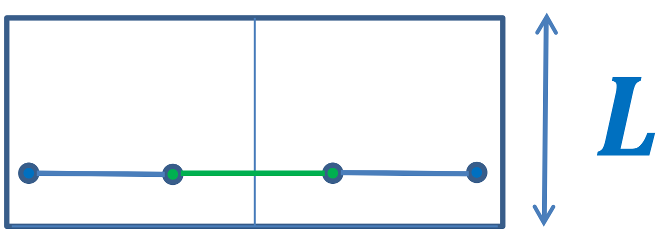
ϵL-nets

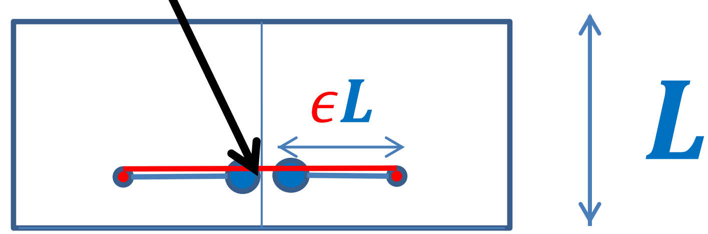
- ϵL-net for a cell C with side length L:
  Collection **S** of vertices in C, every vertex is at distance <= ϵL from some vertex in **S**. (Fact: Can efficiently compute ϵ-net of size $O\left(\frac{1}{\epsilon^2}\right)$)

Bottom-up: For each cell in the quadtree
– Compute optimum MSTs in subcells
– Use ϵL-net from each cell on the next level

- **Idea**: Pay only O(ϵL) for an **edge** cut by cell with side L
- Randomly shift the quadtree:
  $\Pr[cut\ edge\ of\ length\ \ell\ by\ L] \sim \ell/L$ – charge errors

Wrong representative:
O(1)-approximation per level





Randomly shifted quadtree

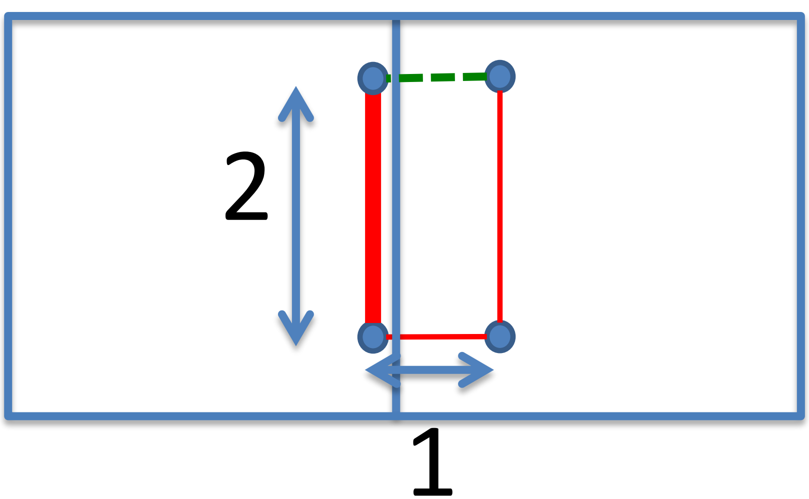
- Top cell shifted by a random vector in $[0, L]^2$

Impose a **randomly shifted** quadtree (top cell length $\mathbf{2\Delta}$)

Bottom-up: For each cell in the quadtree

– Compute optimum MSTs in subcells

– Use ϵL-net from each cell on the next level



Pay **5** instead of **4**

Bad Cut

Pr[Bad Cut] = $\Omega(1)$

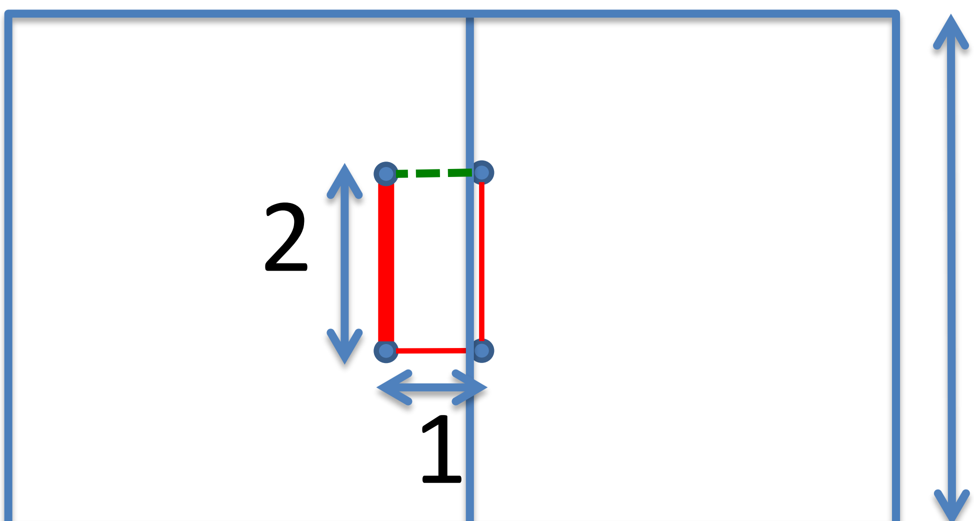
$(1+\epsilon)$-MST in $\mathbf{R} = O(\log\ n)$ rounds

- **Idea:** Only use short edges inside the cells

Impose a **randomly shifted** quadtree (top cell length $\frac{2\Delta}{\epsilon}$)

Bottom-up: For each node (cell) in the quadtree

- compute optimum Minimum Spanning **Forests** in subcells, **using edges of length** $\leq \epsilon L$
- Use only $\epsilon^2 L$-net from each cell on the next level

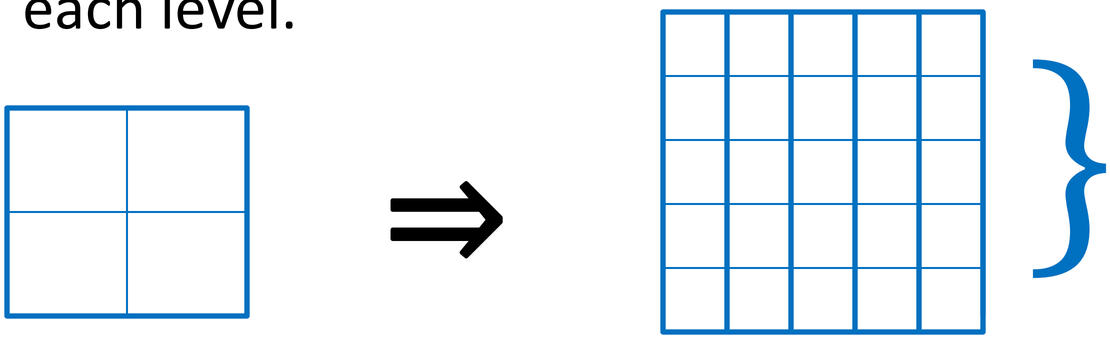


$$L = \Omega(\frac{1}{\epsilon})$$

$$\mathbf{Pr}[\mathbf{Bad\ Cut}] = O(\epsilon)$$

$(1+\epsilon)$-MST in $\mathbf{R} = O(1)$ rounds

- $O(\log n)$ rounds => $O(\log_s n)$ = O(1) rounds
  - Flatten the tree: $(\sqrt{m} \times \sqrt{m})$-grids instead of (2x2) grids at each level.



$\sqrt{m} = n^{\Omega(1)}$

Impose a **randomly shifted** $(\sqrt{m} \times \sqrt{m})$-tree

Bottom-up: For each node (cell) in the tree

- compute optimum MSTs in subcells via edges of length $\leq \epsilon L$
- Use only $\epsilon^2 L$-net from each cell on the next level

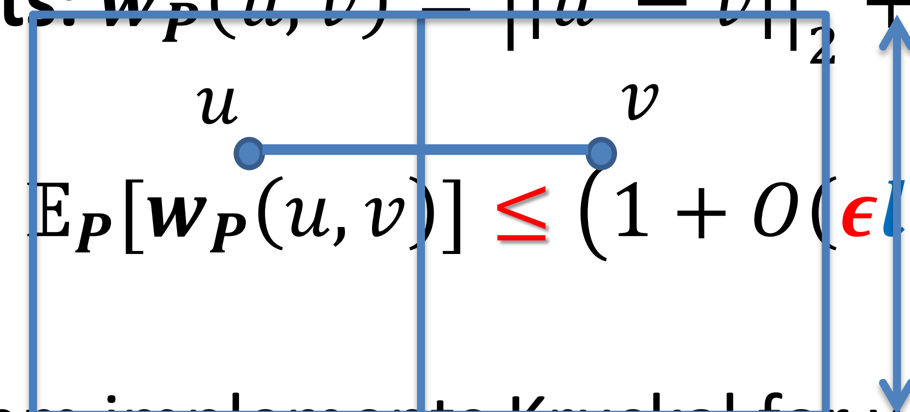
$(1+\epsilon)$-MST in $\mathbf{R} = O(1)$ rounds

Theorem: Let $\boldsymbol{l}$ = # levels in a random tree $\boldsymbol{P}$

$$\mathbb{E}_{\boldsymbol{P}}[\mathbf{ALG}] \leq \left(1 + O(\boldsymbol{\epsilon l d})\right)\mathbf{OPT}$$

Proof (sketch):

- $\boldsymbol{\Delta_P}(u, v)$ = cell length, which first partitions (u, v)
- **New weights:** $\boldsymbol{w_P}(u, v) = \left|\left|u - v\right|\right|_2 + \boldsymbol{\epsilon \Delta_P}(u, v)$



$$\left|\left|u - v\right|\right|_2 \leq \mathbb{E}_{\boldsymbol{P}}[\boldsymbol{w_P}(u, v)] \leq \left(1 + O(\boldsymbol{\epsilon l d})\right)\left|\left|u - v\right|\right|_2$$

- Our algorithm implements Kruskal for weights $\boldsymbol{w_P}$

“Solve-And-Sketch” Framework

$(1+\epsilon)$-**MST**:

- “**Load balancing**”: partition the tree into parts of the same size
- **Almost linear time**: Approximate Nearest Neighbor data structure [Indyk’99]
- Dependence on dimension **d** (size of ϵ-net is $O\left(\frac{d}{\epsilon}\right)^{d}$)
- Generalizes to bounded **doubling dimension**
- Basic version is teachable (Jelani Nelson’s ``**Big Data**’’ class at **Harvard**)
- Implementation in progress…

“Solve-And-Sketch” Framework

$(1+\epsilon)$-**Earth-Mover Distance, Transportation Cost**

- No simple “divide-and-conquer” Arora-Mitchell-style algorithm (unlike for general matching)
- Only recently sequential $(1+\epsilon)$-apprxoimation in $O_\epsilon\left(\boldsymbol{n}\log^{O(1)}\boldsymbol{n}\right)$time [Sharathkumar, Agarwal ‘12]

Our approach (convex sketching):

- Switch to the flow-based version
- In every cell, send the flow to the closest net-point until we can connect the net points

“Solve-And-Sketch” Framework

Convex sketching the cost function for $\boldsymbol{\tau}$ net points

- $F\colon \mathbb{R}^{\boldsymbol{\tau}-1} \to \mathbb{R}$ = the cost of routing fixed amounts of flow through the net points
- Function $F' = F$ + “normalization” is monotone, convex and Lipschitz, $(1+\boldsymbol{\epsilon})$-approximates F
- We can $(1+\boldsymbol{\epsilon})$-sketch it using a lower convex hull

Thank you! http://grigory.us

Open problems:

- Exetension to high dimensions?
  - Probably no, reduce from connectivity => conditional lower bound : $\Omega(\log n)$ rounds for MST in ℓ_∞^n
  - The difficult setting is $d = \Theta(\log \boldsymbol{n})$ (can do JL)
- Streaming alg for **EMD** and **Transporation Cost**?
- Our work:
  - First near-linear time algorithm for **Transportation Cost**
  - Is it possible to reconstruct the solution itself?