Beating the Direct Sum Theorem in Communication Complexity

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Results

Stronger Direct Sum Theorem in communication complexity for equality-type functions

$$R_{\delta}\left(f^{k}\right) \geq \Omega(k) R_{\frac{\delta}{k}}(f)$$

$$D_{\mu^{k},\delta}\left(f^{k}\right) \geq \Omega(k)\, D_{\mu,\frac{\delta}{k}}(f)$$

} **Yao's principle**

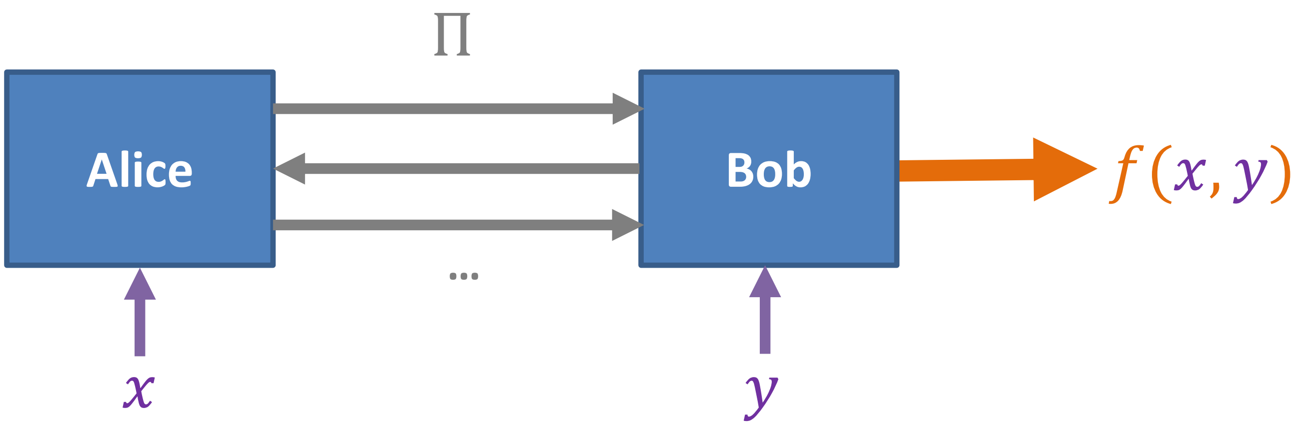
Optimal lower bounds for sketching problems:

- Johnson-Lindenstrauss transform for n vectors
- Pairwise ℓ_1- and ℓ_2-distance estimation
- Matrix multiplication
- Join size estimation of multiple databases

Communication Complexity

- 2 deterministic players: Alice and Bob
- Joint function f
- Communicate and compute

$\Pi(x, y)$ denotes **transcript** or **output**



Communication Complexity

- 2 deterministic players: Alice and Bob
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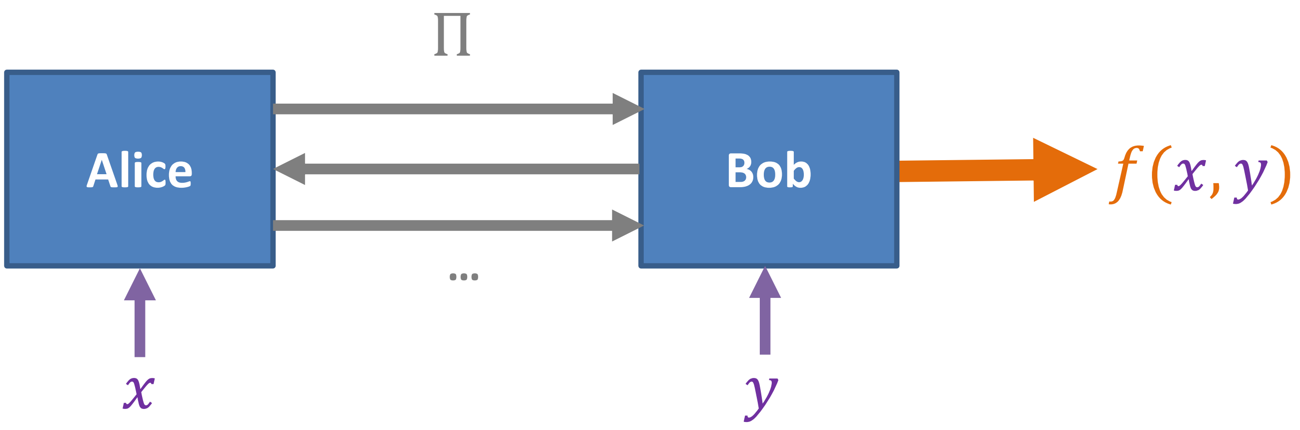
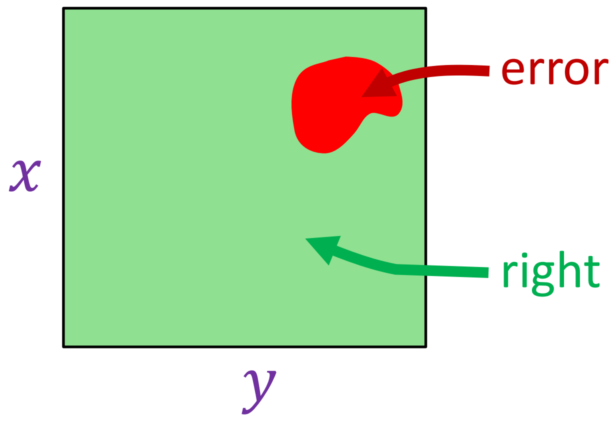


- **Ex:** $x, y \in \{0,1\}^n$, want to output $x \stackrel{?}{=} y$

Communication Complexity

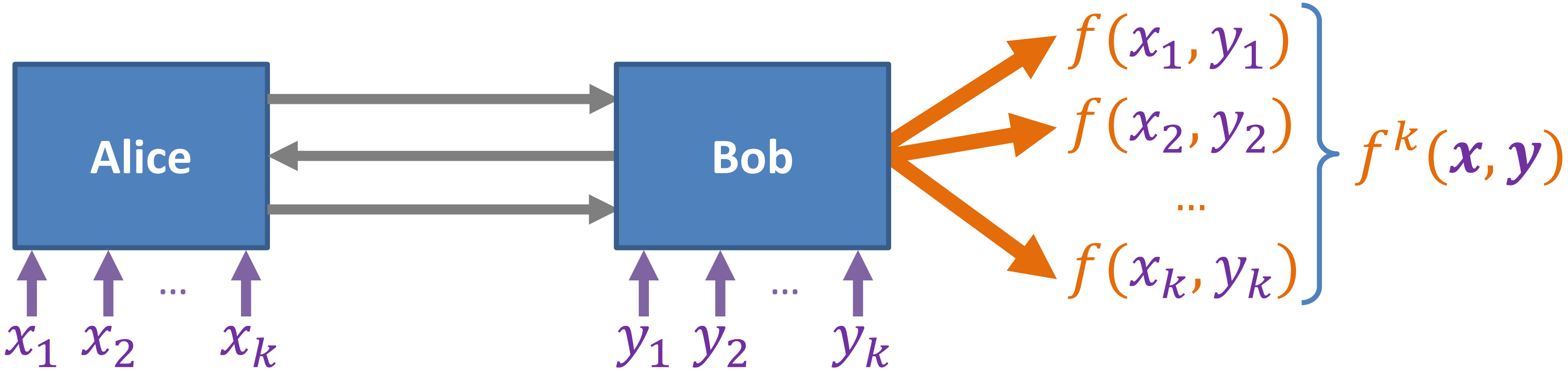
- Consider distribution μ over inputs
- **Goal:** Compute $f(x, y)$ for all but δ μ-fraction of inputs while minimizing longest communication



- **Distributional complexity**

  $D_{\mu,\delta}(f) =$ minimum communication over all δ-protocols

Multiple Instances



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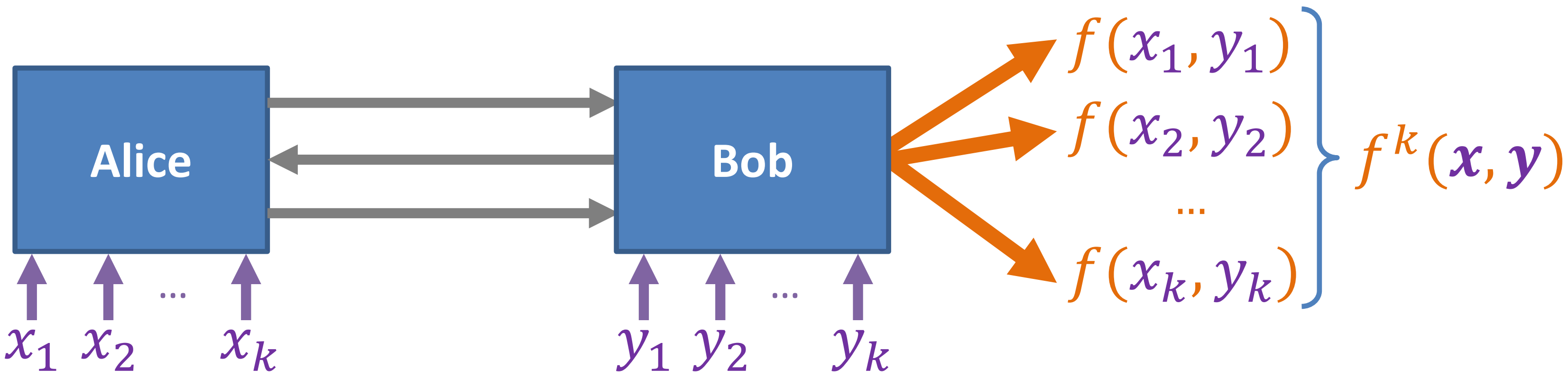


- **Goal:** Compute $f^k(\boldsymbol{x}, \boldsymbol{y})$ for all but δ μ^k-fraction of inputs while minimizing longest communication
- **Distributional complexity:** $D_{\mu^k,\delta}(f^k)$

Multiple Instances

Main question: How much can we save against solving each independently?

$$D_{\mu^k,\delta}(f^k) \overset{?}{\geq} \Omega(k)\, D_{\mu,\frac{\delta}{k}}(f)$$

- **Sometimes a bit:** in the private randomness model

$D_{\mu,\frac{1}{3}}(EQ_n) \geq \Omega(\log n)$ but $D_{\mu,\frac{1}{3}}(EQ_n^n) = O(n)$ [FKNN95]

Multiple Instances

$$D_{\mu^k,\delta}(f^k) \overset{?}{\geq} \Omega(k).D_{\mu,\frac{\delta}{k}}(f)$$

- **Direct sum theorems**
  - D_{μ^k} … [BBCR 10]
  - D_{μ^k} … [BBCR 10]
  - $D^r_{\mu^k,\delta}$ … [BR 11]

None attains above bound

- **Direct product theorems**
  - $D_{\mu^k,1-\left(1-\frac{1}{3}\right)^k}(f^k) \geq \Omega(\sqrt{k}).D_{\mu,\frac{1}{3}}(f)$ [BRWY]

Information Complexity

- **Information cost:** For protocol Π and $(X, Y) \sim \mu$, information revealed about input is

$$\mathrm{IC}_\mu(\Pi) = \mathrm{I}(\Pi(X, Y);\ X, Y) = H(X, Y) - H(X, Y|\Pi)$$

- **Information complexity:**

  $\mathrm{IC}_{\mu,\delta}(f)$ = min information cost over all δ-protocols

Connection: Communication is at least information

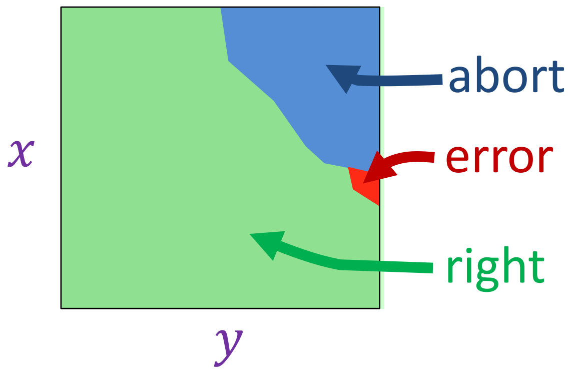
$$D_{\mu,\delta}(f) \geq \mathrm{IC}_{\mu,\delta}(f)$$

- For non-product μ we will work with conditional information complexity $\mathrm{IC}_{\mu,\delta}(f|\nu)$

Protocols with Abortion

Def: A protocol Π $(\boldsymbol{\beta}, \boldsymbol{\delta})$-computes f if

- (Abortion) $\Pr(\Pi(X,Y) = \boldsymbol{abort}) \leq \boldsymbol{\beta}$
- (Error) $\Pr(\Pi(X,Y) \neq f(X,Y) \mid \Pi(X,Y) \neq abort) \leq \boldsymbol{\delta}$



Obs: Stronger guarantee than being wrong with prob. $\approx \boldsymbol{\beta} + \boldsymbol{\delta}$

$\mathrm{IC}_{\mu,\beta,\delta}(f)$ = min information cost over all protocols that $(\boldsymbol{\beta}, \boldsymbol{\delta})$-compute f

Stronger Direct Sum Theorem

Theorem: For every communication problem

Solving k copies with error $\boldsymbol{\delta}$ requires solving each copy with **constant abortion and error $\boldsymbol{\frac{\delta}{k}}$**

Stronger Direct Sum Theorem

- The distribution μ of (X,Y) is a **product** distribution if $\mu(X,Y) = \mu_x(X)\mu_y(Y)$
- $(\mu, \boldsymbol{\nu})$ is a **mixture of product distributions**, if for every $\boldsymbol{t}$ the distribution $(\mu|\boldsymbol{\nu} = \boldsymbol{t})$ is a product distribution

Theorem: For every communication problem f, mixture of product distributions $(\mu, \boldsymbol{\nu})$and $\boldsymbol{\delta} > 0$

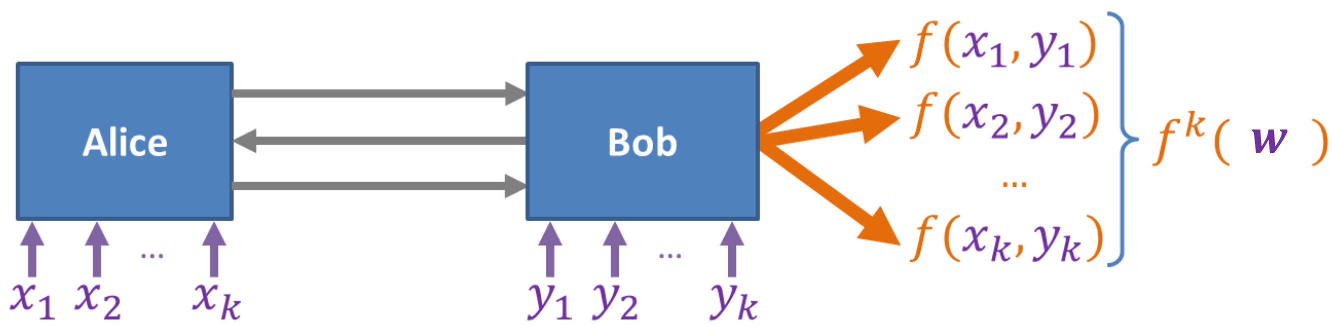
$$\mathrm{IC}_{\mu^{\boldsymbol{k}},\boldsymbol{\delta}}\left(f^{\boldsymbol{k}}|\boldsymbol{\nu}^{\boldsymbol{k}}\right) \geq \Omega(\boldsymbol{k})\ \mathrm{IC}_{\mu,\frac{\boldsymbol{\delta}}{10},O\left(\frac{\boldsymbol{\delta}}{\boldsymbol{k}}\right)}(f|\boldsymbol{\nu})$$

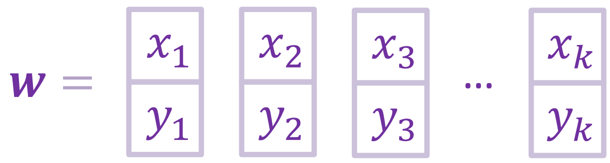
Also holds for one-way and bounded-round communication

Stronger Direct Sum Theorem

Theorem: For every communication problem and product μ

$$\mathrm{IC}_{\mu^k,\delta}(f^k) \geq \Omega(k)\,\mathrm{IC}_{\mu,\frac{\delta}{10},O\left(\frac{\delta}{k}\right)}(f)$$





Stronger Direct Sum Theorem

Proof: Consider protocol Π that computes $f^{\boldsymbol{k}}$ with prob $1-\boldsymbol{\delta}$. Want to show

$$\mathrm{I}(\Pi(\boldsymbol{W});\ \boldsymbol{W}) \geq \Omega(\boldsymbol{k})\mathrm{IC}_{\mu,\frac{\boldsymbol{\delta}}{10},O\left(\frac{\boldsymbol{\delta}}{\boldsymbol{k}}\right)}(f)$$

1) Chain rule:

$$\mathrm{I}(\Pi(\boldsymbol{W}); W) = \sum_{i=1}^{k} \mathrm{I}(\Pi(\boldsymbol{W});\ W_i | \boldsymbol{W}_{<i})$$

By averaging suffices to show that for at least $\Omega(\boldsymbol{k})$ values of $\boldsymbol{i}$

$$\mathrm{I}(\Pi(\boldsymbol{W}); W_i | \boldsymbol{W}_{<i}) \ \geq \mathrm{IC}_{\mu,\frac{\boldsymbol{\delta}}{10},O\left(\frac{\boldsymbol{\delta}}{\boldsymbol{k}}\right)}(f)$$

Want to obtain from Π a protocol **with abortion** to solve i-th copy $f_i^{\boldsymbol{k}}$ with error prob $\frac{\boldsymbol{\delta}}{\boldsymbol{k}}$ and information cost at most

$$\mathrm{I}(\Pi(\boldsymbol{W}); W_i | \boldsymbol{W}_{<i})$$

Stronger Direct Sum Theorem

2) Conditioning amplifies success: For typical i

- $\Pr\left(\Pi_{<i}(\boldsymbol{W}) \neq f_{<i}^{\boldsymbol{k}}(\boldsymbol{W})\right) \leq \boldsymbol{\delta}$
- $\Pr\left(\Pi_i(\boldsymbol{W}) \neq f_i^{\boldsymbol{k}}(\boldsymbol{W}) \mid \Pi_{<i}(\boldsymbol{W}) = f_{<i}^{\boldsymbol{k}}(\boldsymbol{W})\right) = O\left(\frac{\boldsymbol{\delta}}{\boldsymbol{k}}\right)$

This is because

$$1 - \boldsymbol{\delta} \leq \Pr\left(\Pi = f^{\boldsymbol{k}}\right) = \prod_{i=1..k} \Pr\left(\Pi_i = f_i^{\boldsymbol{k}} \mid \Pi_{<i} = f_{<i}^{\boldsymbol{k}}\right)$$

3) Theorem: For a typical i there exists a prefix $\boldsymbol{w}_{<i} \in \mathrm{X}^{\mathrm{i}-1} \times \mathrm{Y}^{\mathrm{i}-1}$ and a set **G** of fixings of the suffix of Π such that:

1. Information cost only changes by a constant factor after fixing $\boldsymbol{w}_{<i}$
2. **G** is a constant fraction of all fixings
3. For every fixing $(\boldsymbol{w}_{<i}, \boldsymbol{w}_{>i} \in \mathbf{G})$ the error probability on W_i is $\leq \frac{\boldsymbol{\delta}}{10}$
4. $\Pr\left(\Pi_i(\boldsymbol{w}_{<i} W_{\boldsymbol{i}} \boldsymbol{w}_{>i}) \neq f_i^{\boldsymbol{k}}(W_{\boldsymbol{i}}) \mid \Pi_{<i}(\boldsymbol{w}_{<i} W_{\boldsymbol{i}} \boldsymbol{w}_{>i}) = f_{<i}^{\boldsymbol{k}}(\boldsymbol{w}_{<i} W_{\boldsymbol{i}} \boldsymbol{w}_{>i})\right) = O\left(\frac{\boldsymbol{\delta}}{\boldsymbol{k}}\right)$

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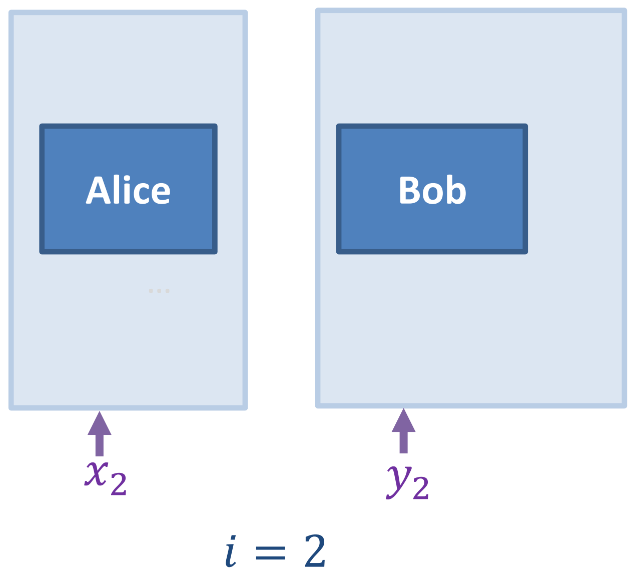
Protocol with abortion for solving f_i^k



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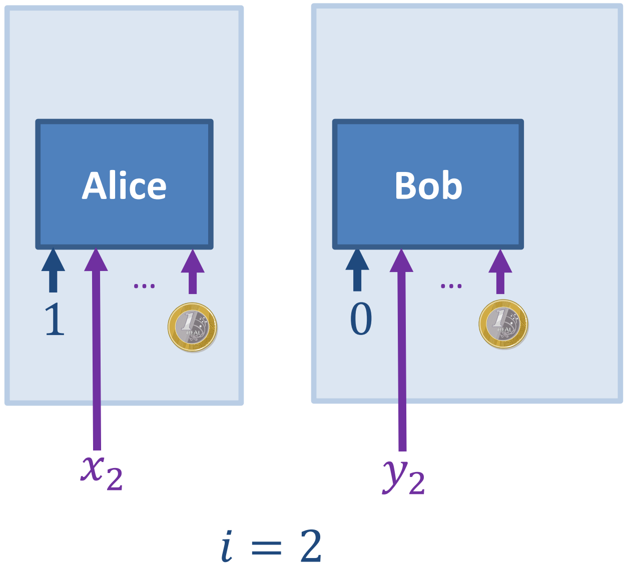
Protocol with abortion for solving f_i^k

- Fix the prefix $\boldsymbol{w}_{<i}$ and sample $\boldsymbol{W}_{>i}$



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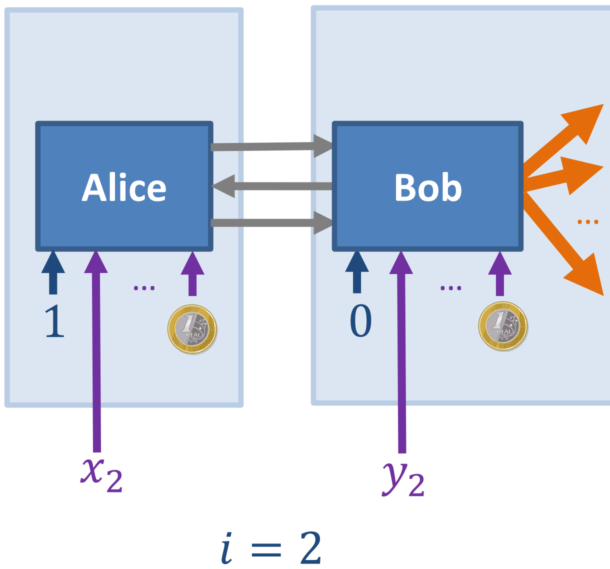
Protocol with abortion for solving f_i^k

- Fix the prefix $\boldsymbol{w}_{<i}$ and sample $\boldsymbol{W}_{>i}$
- Run Π



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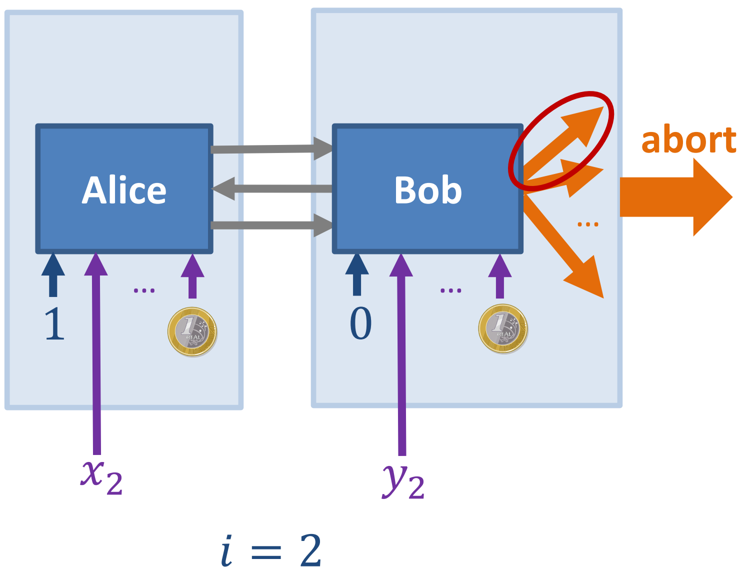
Protocol with abortion for solving f_i^k

- Fix the prefix $\boldsymbol{w}_{<i}$ and sample $\boldsymbol{W}_{>i}$
- Run Π
- Verify if error on some copy $1, 2, \ldots i-1$
  - If so, “abort”



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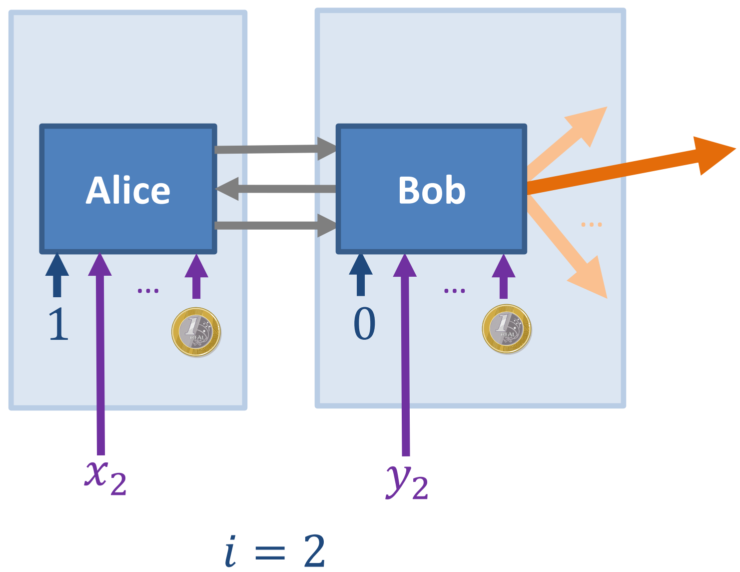
Protocol with abortion for solving f_i^k

- Fix a typical prefix $\boldsymbol{w}_{<i}$ and sample $\boldsymbol{W}_{>i}$
- Run Π
- Verify if error on some copy $1,2,\ldots i-1$
  - If so, "abort"
- Else report i-th output



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Protocol with abortion for solving $f_i^{\boldsymbol{k}}$

- Fix a typical prefix $\boldsymbol{w}_{<i}$ and sample $\boldsymbol{W}_{>i}$
- Run Π
- Verify if error on some copy $1,2,\ldots i-1$
  - If so, "abort"
- Else report i-th output

Protocol $\left(\frac{\delta}{10}, O\left(\frac{\delta}{k}\right)\right)$-computes f_i^k and has information cost exactly $\mathrm{I}(\Pi(\boldsymbol{w}_{<i}\boldsymbol{W}_{\geq \boldsymbol{i}}); W_i)$

Recap

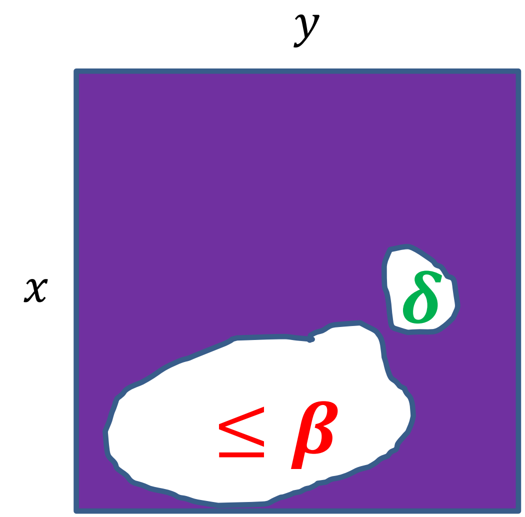
Theorem: For every communication problem

$$\mathrm{IC}_{\mu^{k},\delta}\left(f^{k}|\nu^{k}\right) \geq \Omega(k)\ \mathrm{IC}_{\mu,\frac{1}{20},\frac{\delta}{10},O\left(\frac{\delta}{k}\right)}(f|\nu)$$

Corollary For equality-type problems

$$D_{\mu^{k},\delta}\left(f^{k}\right) \geq \Omega(k)\ \mathrm{IC}_{\mu,\frac{1}{20},\frac{\delta}{10},O\left(\frac{\delta}{k}\right)}(f|\nu) \geq \Omega(k) D_{\mu,\frac{\delta}{k}}(f)$$

Protocols with abortion



- A protocol (α, β, δ)-computes f if with probability $\geq (1-\alpha)$ over its **randomness**
  - It **aborts** with probability $\leq \beta$
  - Conditioned on non-abortion is **correct** w.p. $\geq \mathbf{1-\delta}$

- (μ, ν) is a convex combination of product distributions over $((X \times Y) \times D)$ (marginals: μ over $(X \times Y)$ and ν over D)
- $IC_{\mu,\alpha,\beta,\delta}(f|\nu)$ = minimum information cost of a protocol which (α, β, δ)-computes over (μ, ν).
- $IC_{\mu,\delta}(f|\nu)$ = $IC_{\mu,0,0,\delta}(f|\nu)$

Strong direct sum

- **Strong direct sum:** For every function $\boldsymbol{f}$ and a convex combination of product distributions $(\boldsymbol{\mu}, \boldsymbol{\nu})$

$$\boldsymbol{IC}_{\boldsymbol{\mu}^k,\boldsymbol{\delta}}\left(\boldsymbol{f}^k \middle| \boldsymbol{\nu}^k\right) \geq \Omega(k)\ \boldsymbol{IC}_{\boldsymbol{\mu},\frac{\mathbf{1}}{\mathbf{20}},\frac{\mathbf{1}}{\mathbf{10}},\frac{\boldsymbol{\delta}}{k}}(\boldsymbol{f}|\boldsymbol{\nu})$$

- **Strong** because of high success probability $(1 - \frac{\boldsymbol{\delta}}{k})$
- Gives an extra $\mathbf{log}\,\boldsymbol{k}$ in the lower bound as compared to a weak direct sum [Bar-Yossef, Jayram, Kumar, Sivakumar]

$$\boldsymbol{IC}_{\boldsymbol{\mu}^k,\boldsymbol{\delta}}\left(\boldsymbol{f}^k \middle| \boldsymbol{\nu}^k\right) \geq \Omega(k)\ \boldsymbol{IC}_{\boldsymbol{\mu},\boldsymbol{\delta}}(\boldsymbol{f}|\boldsymbol{\nu})$$

One-way Equality with abortion

- $EQ^{\ell}(\boldsymbol{x},\boldsymbol{y})$ = 1 iff $\boldsymbol{x} = \boldsymbol{y}$, where $\boldsymbol{x}, \boldsymbol{y} \in \{0,1\}^{\ell}$

- **Theorem:** For $\ell = \log(1/20\boldsymbol{\delta})$ there exists $(\boldsymbol{\mu}, \boldsymbol{\nu})$:

$$\boldsymbol{IC}^{\rightarrow}_{\mu,\frac{\mathbf{1}}{\mathbf{20}},\frac{\mathbf{1}}{\mathbf{10}},\delta}\left(EQ^{\ell}\middle|\boldsymbol{\nu}\right) = \boldsymbol{\Omega}(\log(\mathbf{1}/\boldsymbol{\delta}))$$

- **Corollary:** solving $\boldsymbol{k}$ copies of Equality with constant probability requires one-way communication $\boldsymbol{\Omega}(\boldsymbol{k} \log \boldsymbol{k})$ (for sufficiently long strings x_i, y_i)

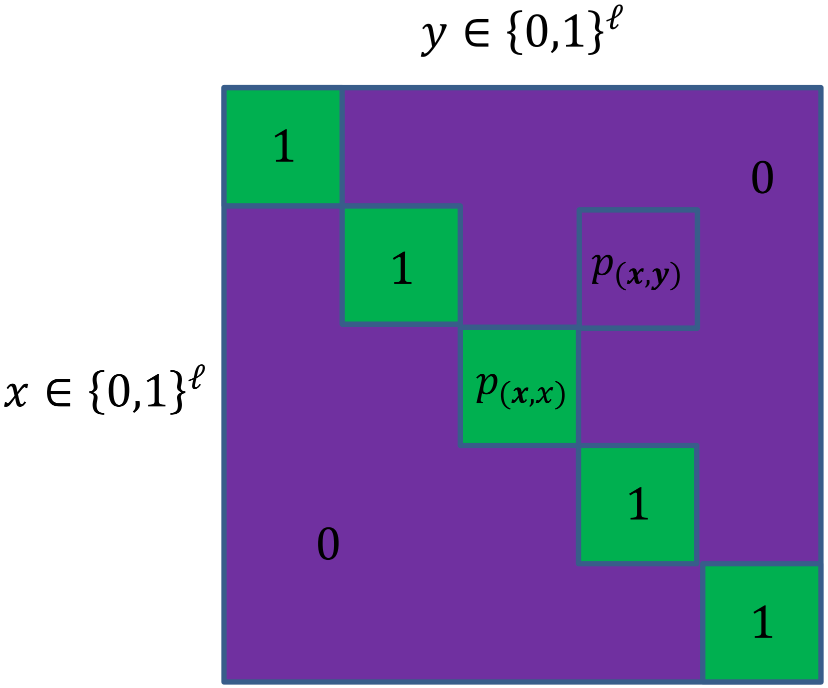
- Hard distribution $((\boldsymbol{X}\ \boldsymbol{Y})\ D_0 \boldsymbol{D})$
  - Random variable for conditioning: $(\mathrm{D}_0 \boldsymbol{D}) \sim U(\{0,1\}^{\ell+1})$
  - If $\mathrm{D}_0 = 0$ then $(\boldsymbol{X}\ \boldsymbol{Y}) \sim U\left(\{0,1\}^{\ell}\right) \times U\left(\{0,1\}^{\ell}\right)$
  - If $\mathrm{D}_0 = 1$ then $\boldsymbol{X} = \boldsymbol{Y} = \boldsymbol{D}$

Equality with abortion

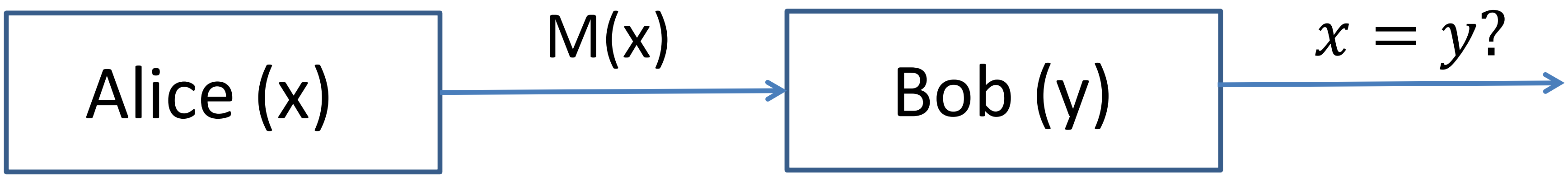
- Hard distribution$((\boldsymbol{X},\boldsymbol{Y}), D_0\boldsymbol{D})$:
  - $(\mathrm{D}_0\boldsymbol{D}) \sim U(\{0,1\}^{\ell+1})$
  - If $\mathrm{D}_0 = 0$ then $(\boldsymbol{X},\boldsymbol{Y}) \sim U(\{0,1\}^{\ell}) \times U(\{0,1\}^{\ell})$
  - If $\mathrm{D}_0 = 1$ then $\boldsymbol{X} = \boldsymbol{Y} = \boldsymbol{D}$ ($\geq \frac{1}{2}$ of the mass on the diagonal)

For $\ell = \log \frac{1}{20\delta}$

- $\boldsymbol{p}_{(\boldsymbol{x},\boldsymbol{x})} = 200\delta^2 + 10\delta$
- $\boldsymbol{p}_{(\boldsymbol{x},\boldsymbol{y})} = 200\delta^2$



Equality with abortion



- $\boldsymbol{IC}^{\rightarrow}_{\boldsymbol{\mu},\ldots}(\boldsymbol{f}|\boldsymbol{\nu}) = \min_{M} I(M(\boldsymbol{X}); \boldsymbol{X}, \boldsymbol{Y}|D_0\boldsymbol{D}) = \min_{M} I(M(\boldsymbol{X}); \boldsymbol{X}|D_0\boldsymbol{D})$
- $I(M(\boldsymbol{X}); \boldsymbol{X}|D_0\boldsymbol{D}) = \boldsymbol{H(X|D_0D)} - \boldsymbol{H(X|M(X), D_0D)}$

- $\boldsymbol{H(X|D_0D)} \geq \Pr[D_0 = 0] \cdot H(\boldsymbol{X}|D_0 = 0, \boldsymbol{D})) = 1/2 \log{(1/20\delta)}$

- By Fano's inequality ([Cover, Thomas]):

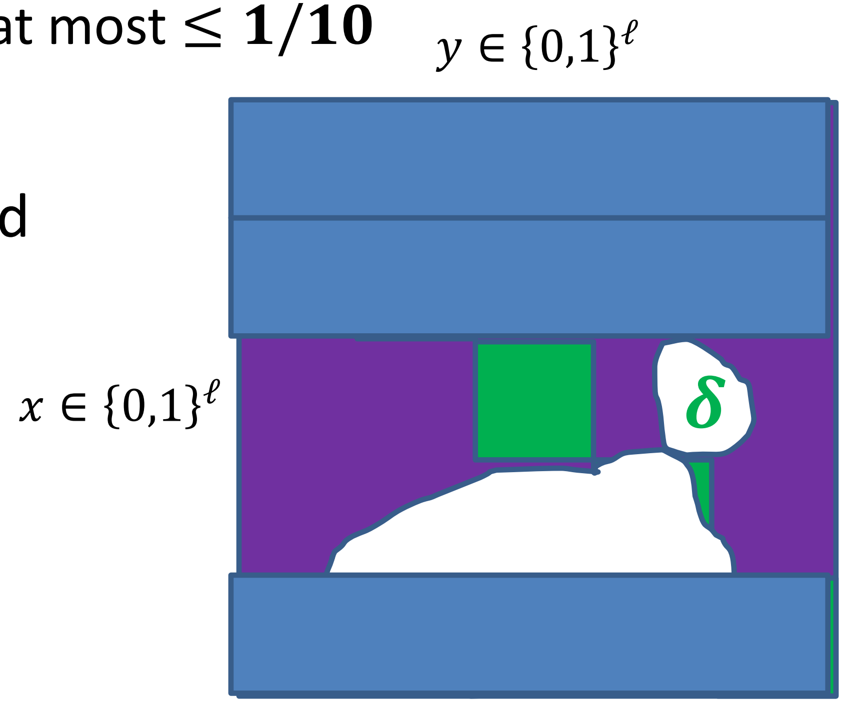
$\boldsymbol{H(X|M(X), D_0D)} \leq 1 + p_e \log(|supp(\boldsymbol{X})|) = 1 + \boldsymbol{p_e} \log{(1/20\delta)}$,

where $\boldsymbol{p_e} = \min_{g} \Pr[g\big(M(\boldsymbol{X}, D_0\boldsymbol{D})\big) \neq \boldsymbol{X}]$ and g is a deterministic function

- Suffices to show that there exists a predictor with error $p_e \leq \frac{2}{5} < \frac{1}{2}$

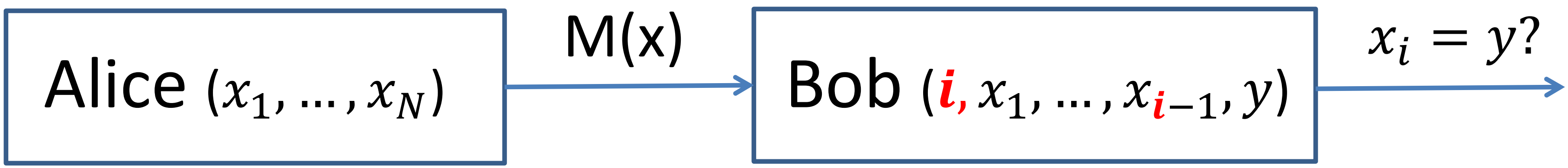
Predictor for Equality

- A row x is **good** if the protocol $\Pi(x, y) = 1$ **iff** $x = y$
- If the $\Pi\left(0, \mathbf{\frac{1}{10}}, \boldsymbol{\delta}\right)$-computes EQ^{ℓ} then $\leq \mathbf{\frac{3}{10}}$ fraction of rows is not **good**:
  - Fraction of rows with an abortion on the diagonal (x, x) is $\leq \mathbf{1/5}$
  - Fraction of rows with an error is at most $\leq \mathbf{1/10}$
- **Predictor**: If the row is **good** then Bob can simulate Π for every y and recover x!
- If $\Pi\left(\mathbf{\frac{1}{20}}\ \mathbf{\frac{1}{10}}, \boldsymbol{\delta}\right)$ -computes EQ^{ℓ} then $p_e \leq \frac{3}{10} + \mathbf{\frac{1}{20}} < \frac{2}{5}$



Augmented indexing

- Augmented indexing over large alphabet ($x_i, y \in [\boldsymbol{m}]$)



- **Theorem:** For sufficiently large $\boldsymbol{m}$ there exists $(\boldsymbol{\mu}, \boldsymbol{\nu})$:
  $$\boldsymbol{IC}^{\rightarrow}_{\boldsymbol{\mu}, \frac{\mathbf{1}}{\mathbf{20}}, \frac{\mathbf{1}}{\mathbf{10}}, \frac{\mathbf{1}}{\boldsymbol{m}}}(Augmented\ Index|\boldsymbol{\nu}) = \boldsymbol{\Omega}(\boldsymbol{N}\ log\ \boldsymbol{m})$$

- **Corollary:** Solving $\boldsymbol{k}$ copies of Augmented Indexing (with const. prob.) requires one-way communication $\boldsymbol{\Omega}(\boldsymbol{N}\ \boldsymbol{k} \log \boldsymbol{k})$ (for sufficiently large alphabet size)

Application: JL-transform of **n** vectors

- Let **S** be a distribution over $\boldsymbol{k} \times d$ matrices, such that for any $v_1, \ldots, v_n \in R^d$ with prob. $\geq 1 - \boldsymbol{\delta}$
$$\left\| \boldsymbol{S} v_i - \boldsymbol{S} v_j \right\|_2 = (1 \pm \boldsymbol{\epsilon}) \left\| v_i - v_j \right\|_2$$
- $\boldsymbol{k} = \#$ rows in $\boldsymbol{S} \geq \frac{\mathbf{1}}{\boldsymbol{\epsilon}^{\mathbf{2}}} \log \left(\frac{\boldsymbol{n}}{\boldsymbol{\delta}} \right)$, dependence on **n** is new
- Even if $\boldsymbol{S}$ is allowed to depend on the first **n**/2 points
- Any enconding $\phi(v_1), \ldots, \phi(v_n)$ that allows pairwise ℓ_p-distance estimation for $p \in \{1,2\}$ requires
$$\Omega\left(\boldsymbol{n}\, \boldsymbol{\epsilon}^{-2} \log \frac{\boldsymbol{n}}{\boldsymbol{\delta}} \, (\log d + \log M) \right) \text{ bits}$$
(M = max abs. value in v_i)

Other applications

- Sketching matrix products
  - Minimum number of columns in a $\boldsymbol{n} \times \boldsymbol{k}$ matrix **S** such that $C = A\boldsymbol{S}\boldsymbol{S}^{\boldsymbol{T}}B$ is a good approximation for AB, where A, B are $\boldsymbol{n} \times \boldsymbol{n}$ matrices?
  - $\left|(AB)_{i,j} - C_{i,j}\right| \leq \epsilon \left|\left|A_i\right|\right|_2 \left|\left|B^j\right|\right|_2$ => $\boldsymbol{k} = O(\epsilon^{-2}\log\frac{\boldsymbol{n}}{\boldsymbol{\delta}})$ [Sarlos]
  - $\left|\left|AB - C\right|\right|_F \leq \epsilon \left|\left|A\right|\right|_F \left|\left|B\right|\right|_F$ => $\boldsymbol{k} = O(\epsilon^{-2}\log\frac{1}{\boldsymbol{\delta}})$ [Clarkson,Woodruff]
  - **Our result:** entry-wise guarantee indeed requires $\boldsymbol{k} = \Omega(\epsilon^{-2}\log\frac{\boldsymbol{n}}{\boldsymbol{\delta}})$

- Optimality of database sketching [Alon, Gibbons, Matias, Szegedy] and mergeable summaries

Open problems

- **Strong direct sum:** For every function $\boldsymbol{f}$ and a convex combination of product distributions $(\boldsymbol{\mu}, \boldsymbol{\nu})$
$$\boldsymbol{IC}_{\mu^k,\delta}\left(\boldsymbol{f}^k \middle| \boldsymbol{\nu}^k\right) \geq \Omega(k)\ \boldsymbol{IC}_{\mu,\frac{1}{20},\frac{1}{10},\frac{\delta}{k}}(\boldsymbol{f}|\boldsymbol{\nu})$$
- More problems with low-error one-way lower bounds?
- Natural problems for low-error 2-way lower bounds (disjointness doesn't work)?
- Applications of direct sums to property testing? [Blais, Brody, Matulef '11, Goldreich '13]
- Strong direct sum for predicates $g\big(f(x_1, y_1), \ldots f(x_k, y_k)\big)$? For OR-EQUALITY ($g = \vee$, $f = EQ^{\ell}$) there is a direct sum [Brody, Chakrabarti, Kondapally'12, Saglam, Tardos'13]