Advances in Linear Sketching over Finite Fields

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CCC'18, with Sampath Kannan (U. Pennsylvania), Elchanan Mossel (MIT) and Swagato Sanyal (NUS)

$\mathbb{F}_2$-Sketching

- Input $\boldsymbol{x} \in \{0,1\}^n$
- Parity = Linear function over $\mathbb{G}F_2$: $\bigoplus_{i \in S} x_i$
- **Deterministic linear sketch**: set of $\boldsymbol{k}$ parities:

$$\ell(\boldsymbol{x}) = \bigoplus_{i_1 \in S_1} x_{i_1}; \bigoplus_{i_2 \in S_2} x_{i_2}; \ldots; \bigoplus_{i_{\boldsymbol{k}} \in S_{\boldsymbol{k}}} x_{i_{\boldsymbol{k}}}$$

- E.g. $x_4 \oplus x_2 \oplus x_{42}$; $x_{239} \oplus x_{30}$; x_{566} ;…
- **Randomized linear sketch**: **distribution** over $\boldsymbol{k}$ parities (random $S_1, S_2, \ldots, S_{\boldsymbol{k}}$):

$$\ell(\boldsymbol{x}) = \bigoplus_{i_1 \in S_1} x_{i_1}; \bigoplus_{i_2 \in S_2} x_{i_2}; \ldots; \bigoplus_{i_{\boldsymbol{k}} \in S_{\boldsymbol{k}}} x_{i_{\boldsymbol{k}}}$$

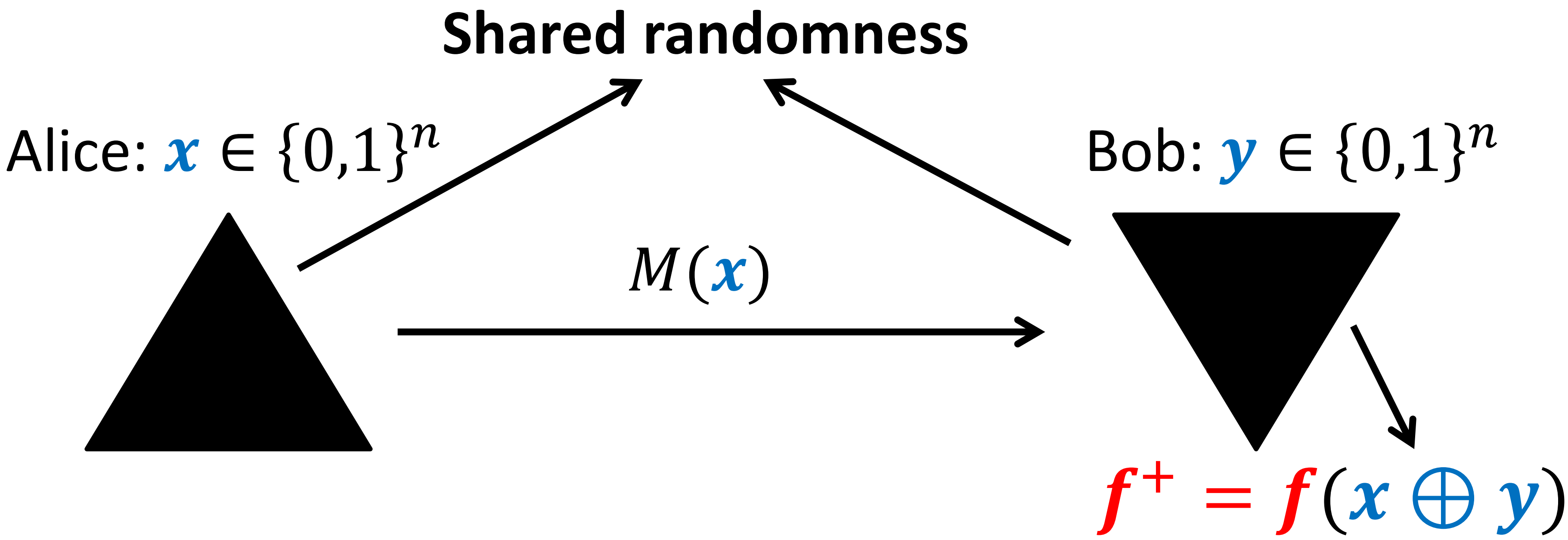
Linear sketching over $\mathbb{F}_2$

- Given $\boldsymbol{f}(\boldsymbol{x}): \{0,1\}^n \to \{0,1\}$
- **Question:**

Can one recover $\boldsymbol{f}(\boldsymbol{x})$ from a small ($\boldsymbol{k} \ll n$) linear sketch over $\mathbb{F}_2$?

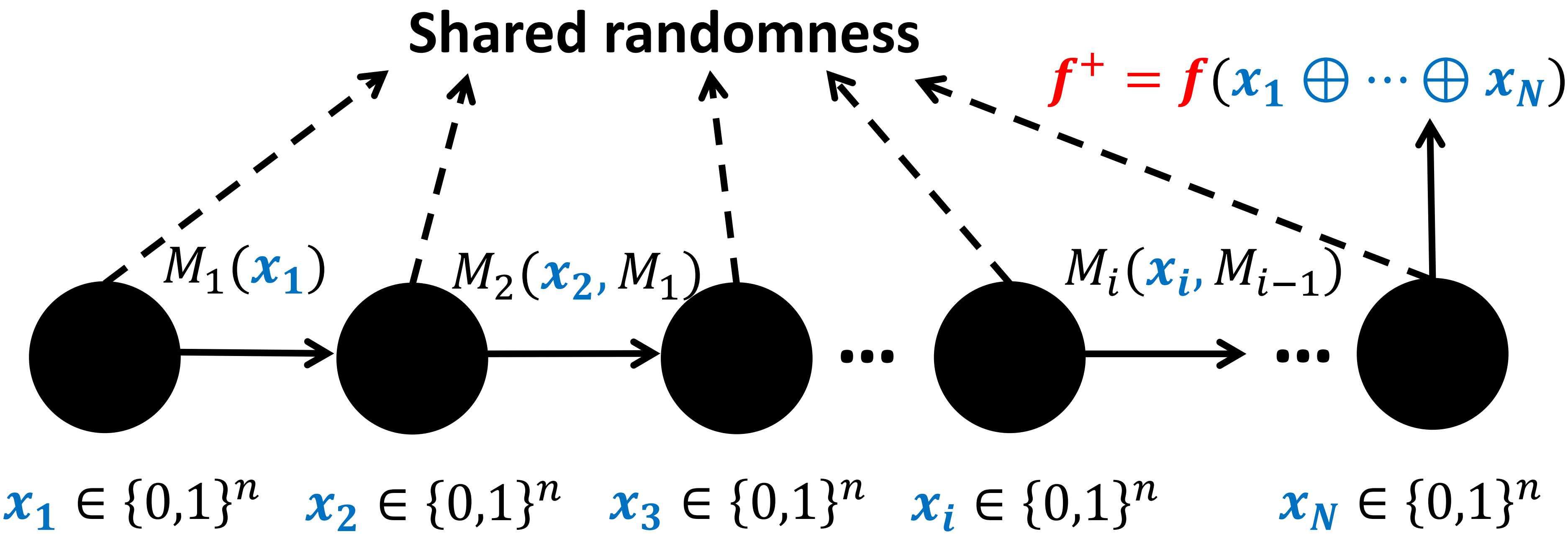
- Allow randomized computation (99% success)
  - Probability over choice of random sets
  - Sets are known at recovery time
  - Recovery is deterministic (w.l.o.g)

Puzzle: Open Problem 78 on Sublinear.info



- **Conjecture**: (Almost) shortest message is a randomized $\mathbb{F}_2$-sketch
- https://sublinear.info/index.php?title=Open_Problems:78

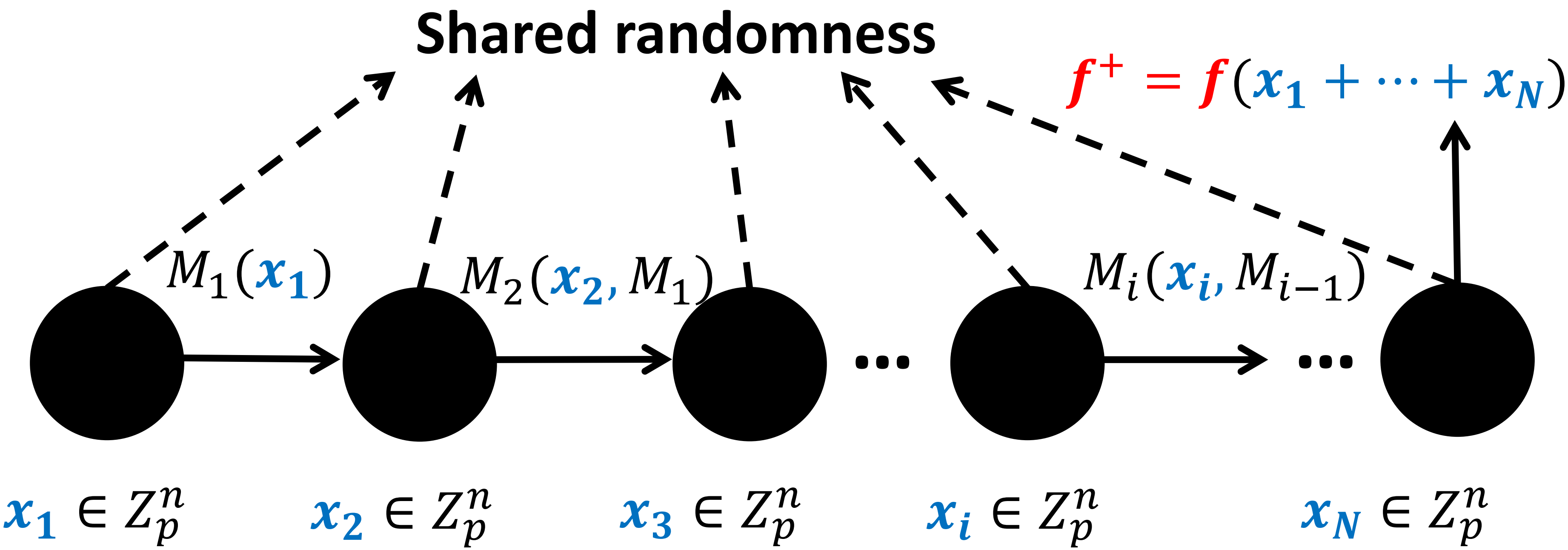
Multi-player version



Thm [Hosseini, Lovett, **Y.**'18; ECCC TR18-169]

For $N \geq 10n$ a protocol where each M_i is at most c bits ⇒ ∃ a randomized $\mathbb{F}_2$-sketch of size $O(c)$

Multi-player version over Z_p



Thm [Hosseini, Lovett, **Y.**'18; ECCC TR18-169]

For $N \geq 10n \log p$ a protocol where each M_i is at most c bits $\Rightarrow \exists$ a randomized Z_p-sketch of dimension $O(c)$

*Holds even for $M_i(x_i, M_1, M_2, \ldots, M_{i-1})$ instead of $M_i(x_i, M_{i-1})$

Motivation: Distributed Computing

- **Distributed computation among M machines:**
  - $\boldsymbol{x} = (\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_M})$ (more generally $\boldsymbol{x} = \bigoplus_{i=1}^{M} \boldsymbol{x_i}$)
  - M machines can compute sketches locally:

$$\ell(\boldsymbol{x_1}), \dots, \ell(\boldsymbol{x_M})$$

  - Send them to the coordinator who computes:

$$\ell_i(\boldsymbol{x}) = \ell_i(\boldsymbol{x_1}) \oplus \cdots \oplus \ell_i(\boldsymbol{x_M})$$ (coordinate-wise XORs)

  - Coordinator computes $\boldsymbol{f}(\boldsymbol{x})$ with $\boldsymbol{kM}$ communication

$\boldsymbol{x}$	0	1	1	1	1	1	0	0	0	0
			$\boldsymbol{x_1}$					$\boldsymbol{x_2}$		

Motivation: Streaming

- $\boldsymbol{x}$ generated through a sequence of updates
- Updates $i_1, \dots, i_m$: update i_t flips bit at position i_t

$\boldsymbol{x}_0$	0	0	0	[illegible]	[illegible]	[illegible]	0	0	0	0

Updates: (1, 3, 8 [illegible]

$\boldsymbol{x}_1$	1	0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	0
$\boldsymbol{x}_2$	1	0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	0
$\boldsymbol{x}_3$	1	0	[illegible]	[illegible]	[illegible]	[illegible]	0	1	0	0
$\boldsymbol{x}$	1	0	0	[illegible]	[illegible]	[illegible]	0	1	0	0

$\ell(\boldsymbol{x})$ allows to recover $\boldsymbol{f}(\boldsymbol{x})$ with $\boldsymbol{k}$ bits of space

Frequently Asked Questions

- **Q:** Why $\mathbb{F}_2$ updates instead of ± 1?
  - Often doesn't help if you know the sign
- **Q:** How to store random sets?
  - Derandomize using Nisan's PRG – extra O(log n) factor in space
- **Q:** Some applications?
  - Essentially all dynamic graph streaming algorithms can be based on L_0-sampling
  - L_0-sampling can be done optimally using $\mathbb{F}_2$-sketching [Kapralov et al. FOCS'17]
- **Q:** Why not allow to compute $\boldsymbol{f}$ approximately?
  - Stay tuned

Deterministic vs. Randomized

- **Fact**: $\boldsymbol{f}$ has a deterministic sketch if and only if
  - $\boldsymbol{f} = \boldsymbol{g}(\bigoplus_{i_1 \in S_1} x_{i_1}; \bigoplus_{i_2 \in S_2} x_{i_2}; \dots; \bigoplus_{i_k \in S_k} x_{i_k})$
  - Equivalent to "$\boldsymbol{f}$ has Fourier dimension $\boldsymbol{k}$"
- **Randomization can help:**
  - **OR**: $\boldsymbol{f}(\boldsymbol{x}) = x_1 \vee \cdots \vee x_n$
  - Has "Fourier dimension" $= n$
  - Pick $\boldsymbol{t} = \log 1/\boldsymbol{\delta}$ random sets $S_1, \dots, S_t$
  - If there is j such that $\bigoplus_{i \in S_j} x_i = 1$ output 1, otherwise output 0
  - Error probability $\boldsymbol{\delta}$

Fourier Analysis

- $\boldsymbol{f}(x_1, \dots, x_n): \{0,1\}^n \to \{0,1\}$
- Notation switch:
  - $0 \to 1$
  - $1 \to -1$
- $\boldsymbol{f}': \{-1,1\}^n \to \{-1,1\}$
- Functions as vectors form a vector space:

$$\boldsymbol{f}: \{-1,1\}^n \to \{-1,1\} \Leftrightarrow \boldsymbol{f} \in \{-1,1\}^{2^n}$$

- Inner product on functions = "correlation":

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle = 2^{-n} \sum_{x \in \{-1,1\}^n} \boldsymbol{f}(x)\boldsymbol{g}(x) = \mathbb{E}_{x \sim \{-1,1\}^n}[\boldsymbol{f}(x)\boldsymbol{g}(x)]$$

$$||\boldsymbol{f}||_2 = \sqrt{\langle f, f \rangle} = \sqrt{\mathbb{E}_{x \sim \{-1,1\}^n}[\boldsymbol{f}^2(x)]} = 1 \text{ (for Boolean only)}$$

"Main Characters" are Parities

- For $S \subseteq [n]$ let **character** $\chi_S(x) = \prod_{i \in S} x_i$
- **Fact**: Every function $f: \{-1,1\}^n \to \{-1,1\}$ is **uniquely** represented as a multilinear polynomial

$$f(x_1, \ldots, x_n) = \sum_{S \subseteq [n]} \hat{f}(S) \chi_S(x)$$

- $\hat{f}(S)$ a.k.a. Fourier coefficient of f on S
- $\hat{f}(S) \equiv \langle f, \chi_S \rangle = \mathbb{E}_{x \sim \{-1,1\}^n}[f(x)\chi_S(x)]$
- $\sum_S \hat{f}(S)^2 = 1$ (Parseval)

Fourier Dimension

- Fourier sets $\boldsymbol{S}$ ≡ vectors in $\mathbb{F}_2^n$
- "$\boldsymbol{f}$ has Fourier dimension $\boldsymbol{k}$" = a $\boldsymbol{k}$-dimensional subspace $A_{\boldsymbol{k}}$ in Fourier domain has all weight

$$\sum_{\boldsymbol{S} \subseteq A_{\boldsymbol{k}}} \hat{\boldsymbol{f}}(\boldsymbol{S})^2 = 1$$

$$\boldsymbol{f}(x_1, \dots, x_n) = \sum_{\boldsymbol{S} \subseteq [n]} \hat{\boldsymbol{f}}(\boldsymbol{S}) \boldsymbol{\chi}_{\boldsymbol{S}}(\boldsymbol{x}) = \sum_{\boldsymbol{S} \subseteq A_{\boldsymbol{k}}} \hat{\boldsymbol{f}}(\boldsymbol{S}) \boldsymbol{\chi}_{\boldsymbol{S}}(\boldsymbol{x})$$

- Pick a basis $\boldsymbol{S_1}, \dots, \boldsymbol{S_k}$ in $A_{\boldsymbol{k}}$:
  - Sketch: $\boldsymbol{\chi}_{\boldsymbol{S_1}}(\boldsymbol{x}), \dots, \boldsymbol{\chi}_{\boldsymbol{S_k}}(\boldsymbol{x})$
  - For every $\boldsymbol{S} \in A_{\boldsymbol{k}}$ there exists $\boldsymbol{Z} \subseteq [\boldsymbol{k}]$: $\boldsymbol{S} = \bigoplus_{\boldsymbol{i} \in \boldsymbol{Z}} \boldsymbol{S_i}$

$$\boldsymbol{\chi}_{\boldsymbol{S}}(\boldsymbol{x}) = \bigoplus_{\boldsymbol{i} \in \boldsymbol{Z}} \boldsymbol{\chi}_{\boldsymbol{S_i}}(\boldsymbol{x})$$

Deterministic Sketching and Noise

Suppose "noise" has bounded norm

$$\boldsymbol{f} = \boldsymbol{g} \oplus \boldsymbol{h} = \boldsymbol{k}\text{-dimensional} \oplus \text{"}\textbf{noise}\text{"}$$

- Sparse Fourier noise (via [Sanyal'15])
  - $\hat{\boldsymbol{f}} = \boldsymbol{k}$-dimensional $\oplus$ "Fourier L_0-noise"
  - $\left\|\widehat{noise}\right\|_0$ = # non-zero Fourier coefficients of noise (aka "Fourier sparsity")
  - Linear sketch size: $\boldsymbol{k} + O(\left\|\widehat{noise}\right\|_0^{1/2})$
  - **Our work**: can't be improved even with randomness and even for uniform x, e.g for ``addressing function''.

How Randomization Handles Noise

- L_0-noise in original domain (via hashing a la OR)
  - $\boldsymbol{f} = \boldsymbol{k}$-dim. $\bigoplus$ "L_0-noise"
  - $\mathbb{F}_2$-sketch size: $\boldsymbol{k}$ + O(log $||noise||_0$)
  - Optimal (but only existentially, i.e. $\exists \boldsymbol{f}$: …)
- L_1-noise in the Fourier domain (via [Bruck, Smolensky '92; Grolmusz'97])
  - $\hat{\boldsymbol{f}} = \boldsymbol{k}$-dim. $\bigoplus$ "Fourier L_1-noise"
  - $\mathbb{F}_2$-sketch size: $\boldsymbol{k} + O(||\widehat{noise}||_1^2)$
  - Example = $\boldsymbol{k}$-dim. $\bigoplus$ small decision tree / DNF / etc.

Randomized Sketching: Hardness

- Not $\boldsymbol{\gamma}$-concentrated on $\boldsymbol{k}$ -dim. Fourier subspaces
  - For $\forall$ $\boldsymbol{k}$ -dim. Fourier subspace A :

$$\sum_{S \notin A} \hat{\boldsymbol{f}}(\boldsymbol{S})^2 \geq 1 - \boldsymbol{\gamma}$$

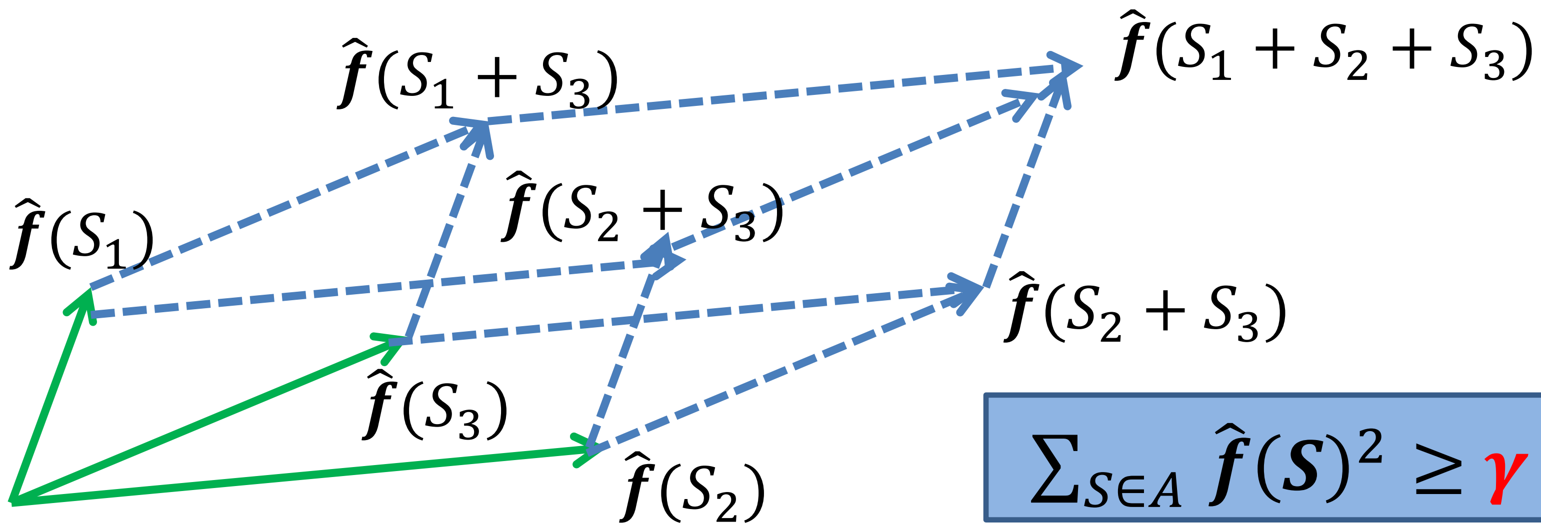
- **Thm.** Any $\boldsymbol{k}$ -dim. $\mathbb{F}_2$-sketch makes error $\frac{1-\sqrt{\boldsymbol{\gamma}}}{2}$
- **Converse doesn't hold, i.e. concentration is not enough**

Randomized Sketching: Hardness

- Not $\boldsymbol{\gamma}$-concentrated on $o(n)$-dim. Fourier subspaces:
  - Almost all **symmetric functions**, i.e. $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{h}(\sum_i x_i)$
    - If not Fourier-close to constant or $\bigoplus_{i=1}^{n} x_i$
    - E.g. Majority (not an extractor even for $O(\sqrt{n})$)
  - **Tribes** (balanced DNF)
  - **Recursive majority**: $Maj^{\circ k} = Maj_3 \circ Maj_3 \ldots \circ Maj_3$

Approximate Fourier Dimension

- Not γ-concentrated on k -dim. Fourier subspaces
  - $\forall$ k -dim. Fourier subspace A: $\sum_{S \notin A} \hat{f}(S)^2 \geq 1 - \gamma$
  - Any k -dim. linear sketch makes error ½$(1 - \sqrt{\gamma})$
- **Definition** (Approximate Fourier Dimension)
  - $\dim_{\gamma}(f)$ $=$ smallest d such that f is γ-concentrated on some Fourier subspace of dimension d



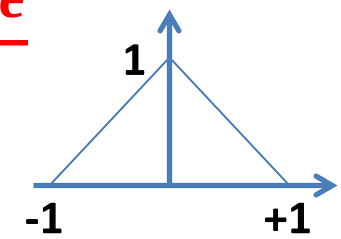
$\sum_{S \in A} \hat{f}(S)^2 \geq \gamma$

Sketching over Uniform Distribution + Approximate Fourier Dimension

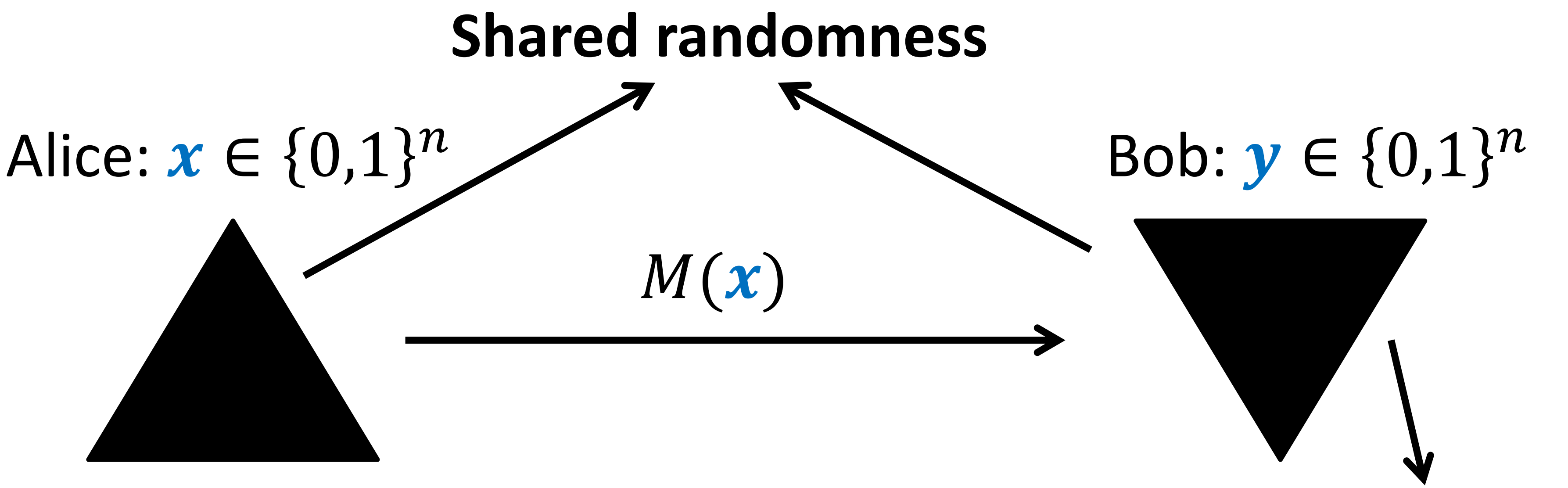
- Sketching error over **uniform distribution of $\boldsymbol{x}$**.
- $\dim_{\epsilon}(\boldsymbol{f})$-dimensional sketch gives error $\mathbf{1-\epsilon}$:
  - Fix $\dim_{\epsilon}(\boldsymbol{f})$-dimensional A: $\sum_{S \in A} \hat{\boldsymbol{f}}(\boldsymbol{S})^2 \geq \boldsymbol{\epsilon}$
  - Output: $\boldsymbol{g}(x) = \mathrm{sign}\left(\sum_{\boldsymbol{S} \in A} \hat{\boldsymbol{f}}(\boldsymbol{S}) \chi_{\boldsymbol{S}}(x)\right)$:

$$\Pr_{x \sim U(\{-1,1\}^n)}[\boldsymbol{g}(x) = \boldsymbol{f}(x)] \geq \boldsymbol{\epsilon} \Rightarrow \text{error } \mathbf{1-\epsilon}$$

- We show a basic refinement $\Rightarrow$ error $\frac{\mathbf{1-\epsilon}}{\mathbf{2}}$
  - Pick $\boldsymbol{\theta}$ from a carefully chosen distribution
  - Output: $\boldsymbol{g}_{\boldsymbol{\theta}}(x) = \mathrm{sign}\left(\sum_{\boldsymbol{S} \in A} \hat{\boldsymbol{f}}(\boldsymbol{S}) \chi_{\boldsymbol{S}}(x) - \boldsymbol{\theta}\right)$



1-way Communication Complexity of XOR-functions



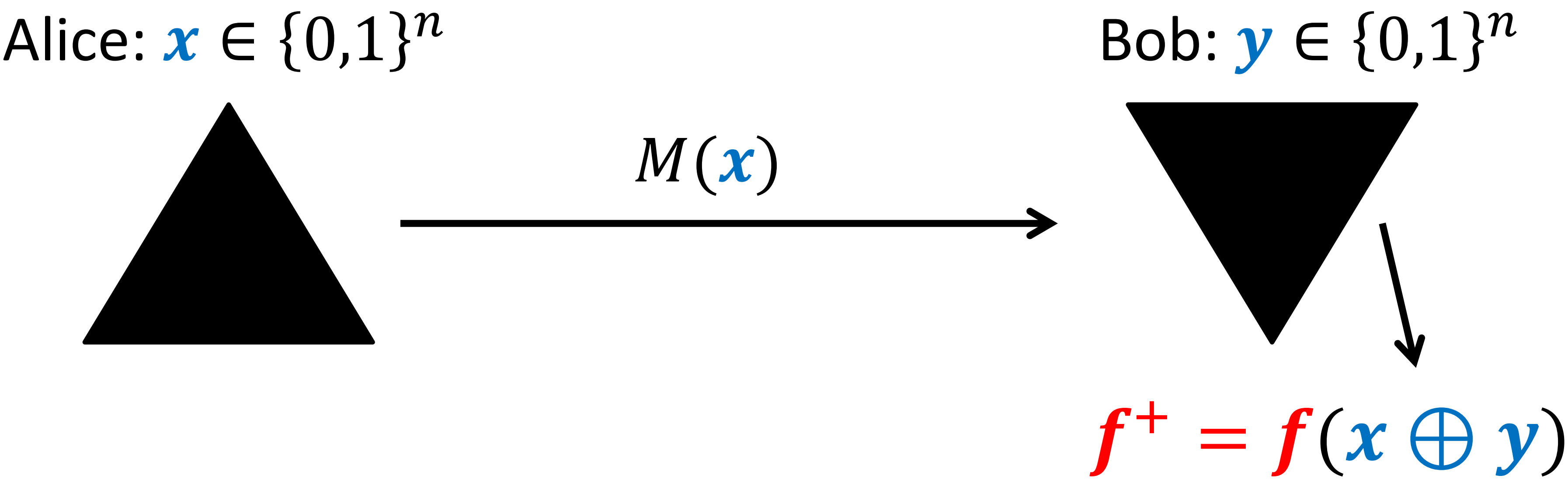
$\boldsymbol{f}^+ = \boldsymbol{f}(\boldsymbol{x} \oplus \boldsymbol{y})$

- Examples:
  - $\boldsymbol{f}(\mathrm{z}) = OR_{i=1}^{n}(z_i) \Rightarrow \boldsymbol{f}^+$: (not) Equality
  - $\boldsymbol{f}(\mathrm{z}) = (\left||z|\right|_0 > \mathrm{d}) \Rightarrow \boldsymbol{f}^+$: Hamming Dist $> \mathrm{d}$
- $R_\epsilon^1(\boldsymbol{f}^+)$ = min.|M| so that Bob's error prob. ϵ

Communication Complexity of XOR-functions

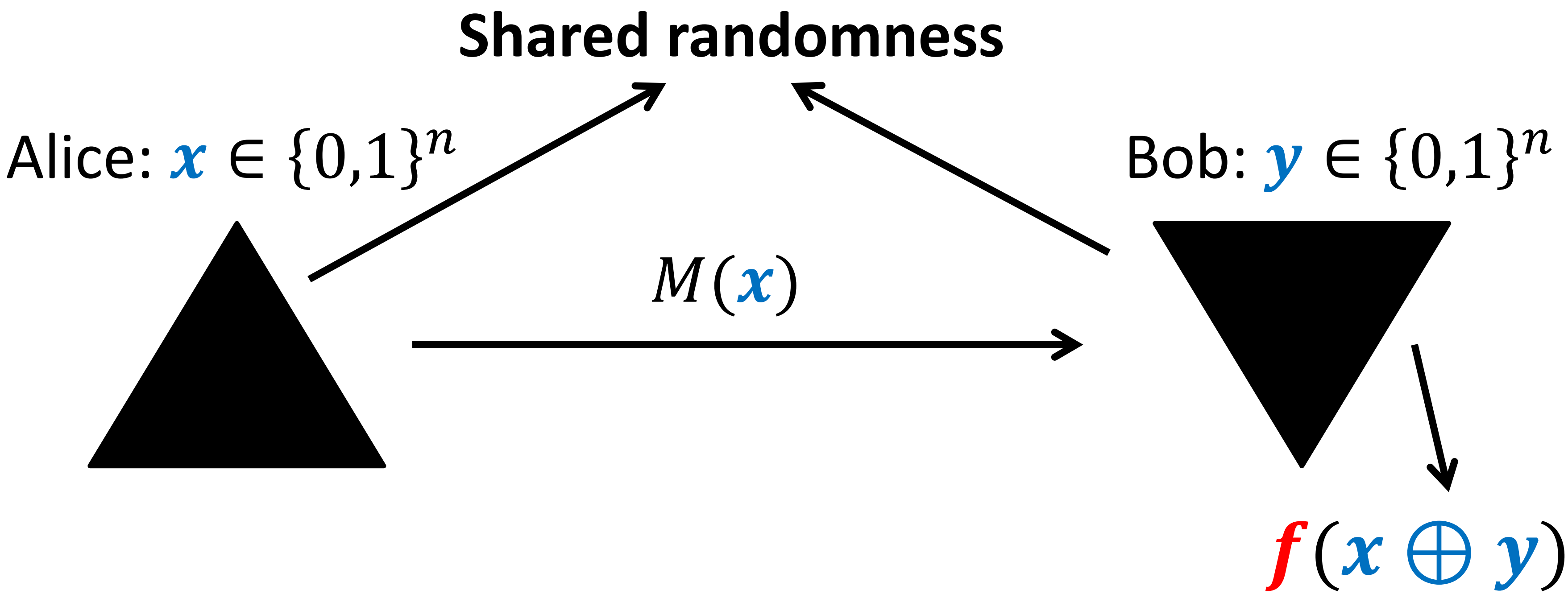
- Well-studied (often for 2-way communication):
  - [Montanaro,Osborne], ArXiv'09
  - [Shi, Zhang], QIC'09,
  - [Tsang, Wong, Xie, Zhang], FOCS'13
  - [O'Donnell, Wright, Zhao,Sun,Tan], CCC'14
  - [Hatami, Hosseini, Lovett], FOCS'16
- Connections to log-rank conjecture [Lovett'14]:
  - Even special case for XOR-functions still open

Deterministic 1-way Communication Complexity of XOR-functions



- $D^1(\boldsymbol{f}^+)$ = min. |M| so that Bob is always correct
- [Montanaro-Osborne'09]: $D^1(\boldsymbol{f}) = D^{lin}(\boldsymbol{f})$
- $D^{lin}(\boldsymbol{f})$ = deterministic $\mathbb{F}_2$-sketch complexity of $\boldsymbol{f}^+$
- $D^1(\boldsymbol{f}^+) = D^{lin}(\boldsymbol{f})$ = Fourier dimension of $\boldsymbol{f}$

1-way Communication Complexity of XOR-functions



- $R_\epsilon^1(\boldsymbol{f})$ = min.|M| so that Bob's error prob. ϵ
- $R_\epsilon^{lin}(\boldsymbol{f}^+) =$ rand. $\mathbb{F}_2$-sketch complexity (error ϵ)
- $R_\epsilon^1(\boldsymbol{f}^+) \leq R_\epsilon^{lin}(\boldsymbol{f})$
- **Conjecture:** $R_\epsilon^1(\boldsymbol{f}^+) \approx R_\epsilon^{lin}(\boldsymbol{f})$?

$$R^{1}_{\epsilon}(\boldsymbol{f}^{+}) \approx R^{lin}_{\epsilon}(\boldsymbol{f})?$$

As we show holds for:

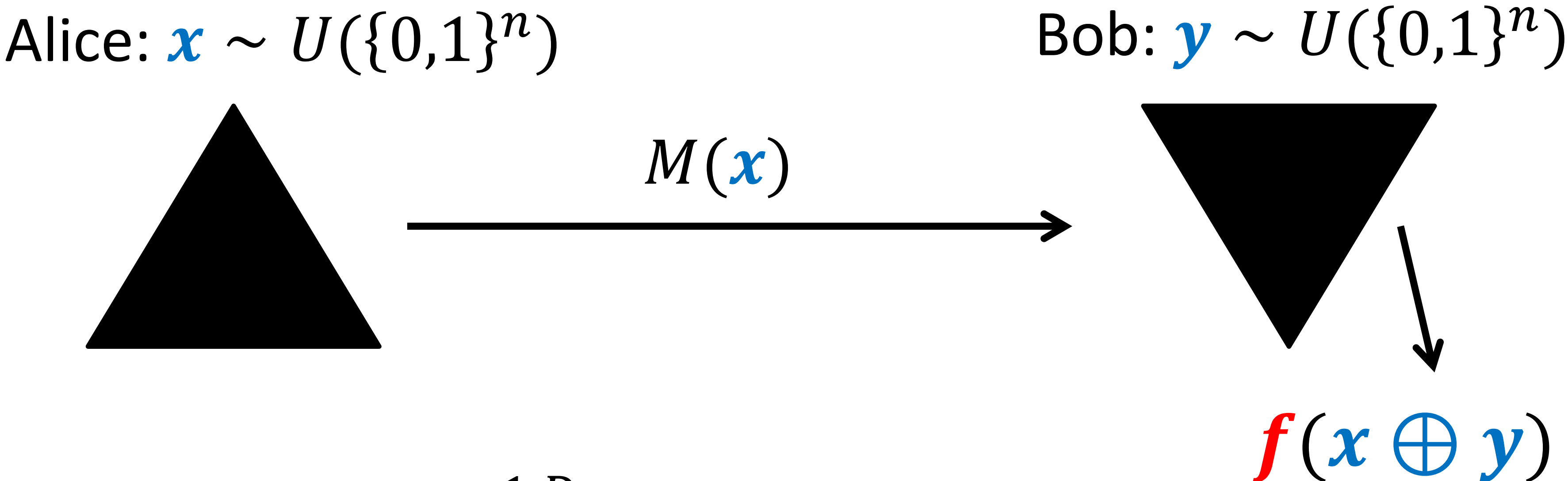
- Majority, Tribes, recursive majority, addressing function
- (Almost all) symmetric functions
- Degree-$\boldsymbol{d}$ $\mathbb{F}_2$-polynomials:

$$R^{lin}_{5\epsilon}(\boldsymbol{f}) = O(\boldsymbol{d}\, R^{1}_{\epsilon}(\boldsymbol{f}^{+}))$$

Analogous question for 2-way is wide open:

[HHL'16] $Q^{\oplus-dt}_{\epsilon}(\boldsymbol{f}) = poly(R_{\epsilon}\ (\boldsymbol{f}^{+}))$?

Distributional 1-way Communication under Uniform Distribution



- $R_\epsilon^1(\boldsymbol{f}) = \sup_D \mathfrak{D}_\epsilon^{1,D}(\boldsymbol{f})$
- $\mathfrak{D}_\epsilon^{1,U}(\boldsymbol{f})$ = min.|M| so that Bob's error prob. ϵ is over the uniform distribution over $(\boldsymbol{x}, \boldsymbol{y})$
- Enough to consider deterministic messages only
- Motivation: streaming/distributed with random input

Communication for Uniform Distribution

Thm: If $\dim_{\epsilon}(\boldsymbol{f}) = \boldsymbol{d} - 1$ then $\mathfrak{D}^{1,U}_{\frac{1-\epsilon}{6}}(\boldsymbol{f}^{+}) \geq \frac{\boldsymbol{d}}{6}$.

- Optimal up to constant factors in dimension and error
  - $\boldsymbol{d}$-dim. linear sketch has error $\frac{1-\epsilon}{2}$

Application: Random Streams

- $\boldsymbol{x} \in \{0,1\}^n$ generated via a stream of updates
  - Each update flips a **random coordinate**

- **Goal**: maintain $\boldsymbol{f}(\boldsymbol{x})$ during the stream (error prob. $\boldsymbol{\epsilon}$)
- **Question**: how much space necessary?
- **Answer:** $\mathfrak{D}_{\epsilon}^{1,U}$ and best algorithm is linear sketch
  - After first O($n \log n$) updates input $\boldsymbol{x}$ is uniform

- **Big open question:**
  - Is the same true if $\boldsymbol{x}$ is not uniform?
  - True for **VERY LONG** ($2^{2^{2^{\Omega(n)}}}$) streams (via [LNW'14])
  - True for streams of length ($O(n^2)$) (via [HLY'18])

Approximate $\mathbb{F}_2$-Sketching [Y.'17]

- $\boldsymbol{f}(x_1, \ldots, x_n): \{0,1\}^n \to \mathbb{R}$
- Normalize: $||\boldsymbol{f}||_{\mathbf{2}} = 1$
- **Question:**

Can one compute $\boldsymbol{f}'$: $\mathbb{E}[\left(\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}'(\boldsymbol{x})\right)^{\mathbf{2}} \leq \boldsymbol{\epsilon}]$ from a small ($\boldsymbol{k} \ll n$) linear sketch over $\mathbb{F}_2$?

Approximate $\mathbb{F}_2$-Sketching [**Y.**'17]

Interesting facts:

- All results under the **uniform distribution** generalize directly to approximate sketching
- L_1-sampling has optimal dependence on parameters:
  – Optimal dependence: $O\left(\frac{\left\|\hat{f}\right\|_1^2}{\epsilon}\right)$
  – **Open:** Is L_1-sampling optimal for Boolean functions?

Approximate $\mathbb{F}_2$-Sketching of Valuation Functions [**Y.**,Zhou'18]

- Additive ($\sum_{i=1}^{n} w_i x_i$):
  - $\Theta\left(\min\left(\frac{||w||_1^2}{\epsilon}, n\right)\right)$ (optimal via weighted Gap Hamming)
- Budget-additive ($\min(\mathrm{b}, \sum_{i=1}^{n} w_i x_i)$):
  - $\Theta\left(\min\left(\frac{||w||_1^2}{\epsilon}, n\right)\right)$
- Coverage:
  - Optimal $\Theta\left(\frac{1}{\epsilon}\right)$ (via L_1-Sampling)
- Matroid rank (various results depending on rank r)
- α-Lipschitz submodular functions:
  - $\Omega(n)$ communication lower bound for $\alpha = \Omega(1/n)$
  - Uses a large family of matroids from [Balcan, Harvey'10]

Thanks! Questions?

- Other stuff [**Y.**, Zhou 18]:
  - Linear Threshold Functions: $\Theta\left(\frac{\theta}{m}\log\frac{\theta}{m}\right)$
    - Resolves a communication conjecture of [MO'09]
  - Simple neural nets: LTF(ORs), LTF(LTFs)
- Blog post: http://grigory.us/blog/the-binary-sketchman

Sketching over Uniform Distribution

Thm: If $\dim_{\epsilon}(f) = d - 1$ then $\mathfrak{D}^{1,U}_{\frac{1-\epsilon}{6}}(f^+) \geq \frac{d}{6}$.

- Optimal up to error as d-dim. linear sketch has error $\frac{1-\epsilon}{2}$

Weaker: If $\epsilon_2 > \epsilon_1$, $\dim_{\epsilon_1}(f) = \dim_{\epsilon_2}(f) = d - 1$ then:

$$\mathfrak{D}^{1,U}_{\delta}(f) \geq d,$$

where $\delta = (\epsilon_2 - \epsilon_1)/4$.

Corollary: If $\hat{f}(\emptyset) < C$ for $C < 1$ then there exists d:

$$\mathfrak{D}^{1,U}_{\Theta\left(\frac{1}{n}\right)}(f) \geq d.$$

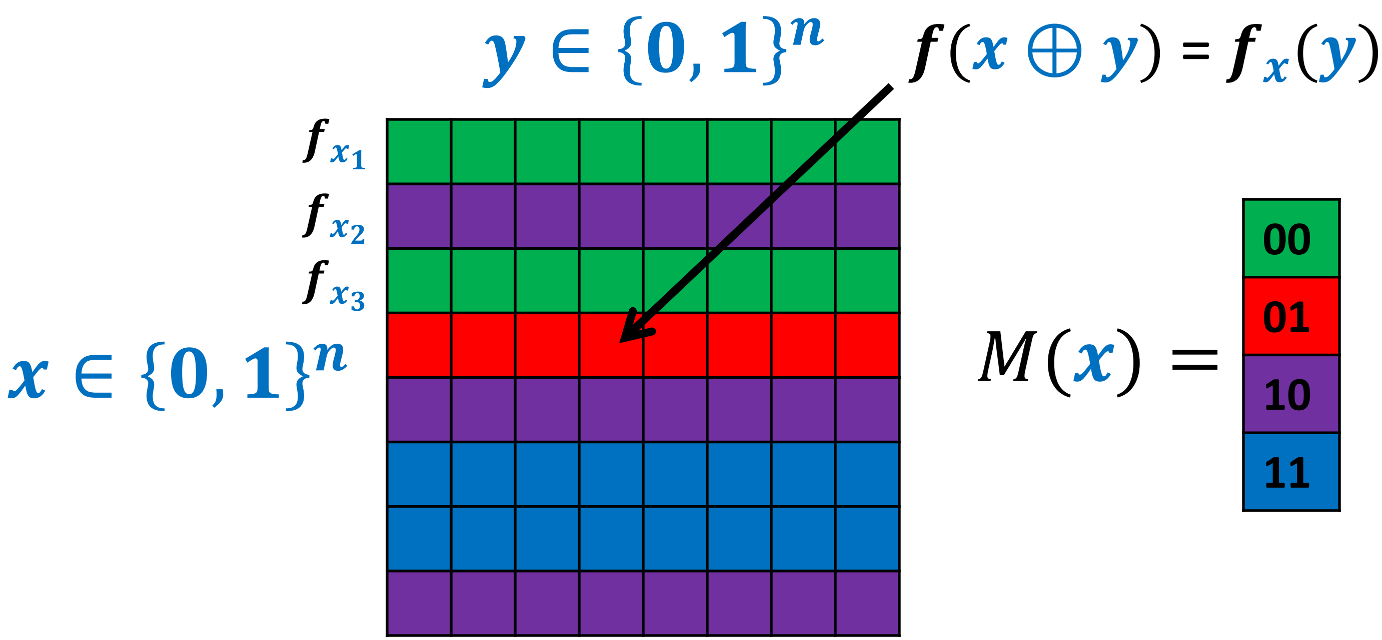
- Tight for the Majority function, etc.

$\mathfrak{D}_{\epsilon}^{1,U}$ and Approximate Fourier Dimension

Thm: If $\epsilon_2 > \epsilon_1 > 0, \dim_{\epsilon_1}(f) = \dim_{\epsilon_2}(f) = d - 1$ then:

$$\mathfrak{D}_{\delta}^{1,U}(f) \geq d,$$

where $\delta = (\epsilon_2 - \epsilon_1)/4$.



$\mathfrak{D}_{\epsilon}^{1,U}$ and Approximate Fourier Dimension

- If $|\boldsymbol{M}(\boldsymbol{x})| = \boldsymbol{d} - 1$ average "rectangle" size = $2^{\boldsymbol{n}-\boldsymbol{d}+\boldsymbol{1}}$
- A subspace A **distinguishes** $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ if:

$$\exists \boldsymbol{S} \in A : \chi_{\boldsymbol{S}}(\boldsymbol{x}_1) \neq \chi_{\boldsymbol{S}}(\boldsymbol{x}_2)$$

- **Lem 1:** Fix a $\boldsymbol{d}$-dim. subspace $A_{\boldsymbol{d}}$: typical $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in a typical "rectangle" are distinguished by $A_{\boldsymbol{d}}$
- **Lem 2:** If a $\boldsymbol{d}$-dim. subspace $A_{\boldsymbol{d}}$ distinguishes $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ +

1) $\boldsymbol{f}$ is $\boldsymbol{\epsilon}_2$-concentrated on $A_{\boldsymbol{d}}$
2) $\boldsymbol{f}$ is not $\boldsymbol{\epsilon}_1$-concentrated on any $(\boldsymbol{d} - 1)$-dim. subspace

$$\Rightarrow \Pr_{z \sim U(\{-1,1\}^n)}\left[\boldsymbol{f}_{\boldsymbol{x}_1}(z) \neq \boldsymbol{f}_{\boldsymbol{x}_2}(z)\right] \geq \boldsymbol{\epsilon}_2 - \boldsymbol{\epsilon}_1$$

$\mathfrak{D}_{\epsilon}^{1,U}$ and Approximate Fourier Dimension

Thm: If $\epsilon_2 > \epsilon_1 > 0, \dim_{\epsilon_1}(f) = \dim_{\epsilon_2}(f) = d - 1$ then:

$$\mathfrak{D}_{\delta}^{1,U}(f) \geq d,$$

Where $\delta = (\epsilon_2 - \epsilon_1)/4$.

$$\Pr_{z \sim U(\{-1,1\}^n)}[f_{x_1}(z) \neq f_{x_2}(z)] \geq \epsilon_2 - \epsilon_1$$

				y				
g_{x_1}	0	1	1	0	0	0	1	0
g_{x_2}	0	1	0	1	0	1	1	0
				0				

R = "typical rectangle"

Error for fixed y = $\min(\Pr_{x \in R}[f_x(y) = 0], \Pr_{x \in R}[f_x(y) = 1])$

Average error for $(x, y) \in R$ = $\Omega(\epsilon_2 - \epsilon_1)$

Example: Majority

- Majority function:

$$\boldsymbol{Maj_n}(z_1, \dots, z_n) \equiv \sum_{i=1}^{n} z_i \geq n/2$$

- $\widehat{\boldsymbol{Maj_n}}(\boldsymbol{S})$ only depends on $|\boldsymbol{S}|$
- $\widehat{\boldsymbol{Maj_n}}(\boldsymbol{S}) = 0$ if $|\boldsymbol{S}|$ is odd
- $W^{\boldsymbol{k}}(\boldsymbol{Maj_n}) = \sum_{\boldsymbol{S}:|\boldsymbol{S}|=\boldsymbol{k}} \widehat{\boldsymbol{Maj_n}}(\boldsymbol{S}) = \alpha \boldsymbol{k}^{-\frac{3}{2}}\left(1 \pm O\left(\frac{1}{\boldsymbol{k}}\right)\right)$
- $(n-1)$-dimensional subspace with most weight:

$$\boldsymbol{A_{n-1}} = span(\{1\},\{2\},\dots,\{n-1\})$$

- $\sum_{\boldsymbol{S} \in \boldsymbol{A_{n-1}}} \widehat{\boldsymbol{Maj_n}}(\boldsymbol{S}) = 1 - \frac{\gamma}{\sqrt{n}} \pm O\left(n^{-3/2}\right)$
- Set $\boldsymbol{\epsilon_2} = 1 - O(n^{-3/2})$, $\boldsymbol{\epsilon_1} = 1 - \frac{\gamma}{\sqrt{n}} + O\left(n^{-3/2}\right)$

$$\mathfrak{D}^{1,U}_{O(1/\sqrt{n})}(\boldsymbol{Maj_n}) \geq \boldsymbol{n}$$