Property Testing and Communication Complexity

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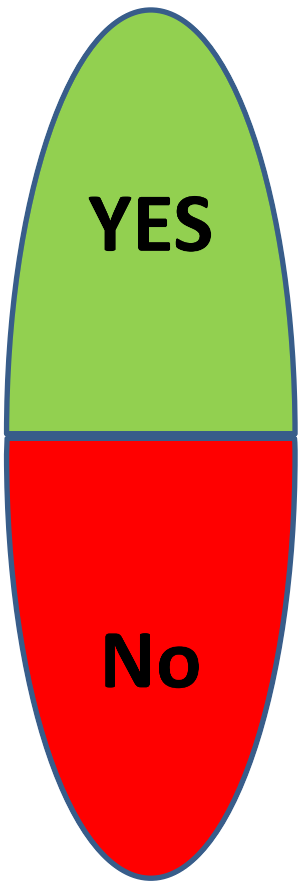
Property Testing

[Goldreich, Goldwasser, Ron, Rubinfeld, Sudan]

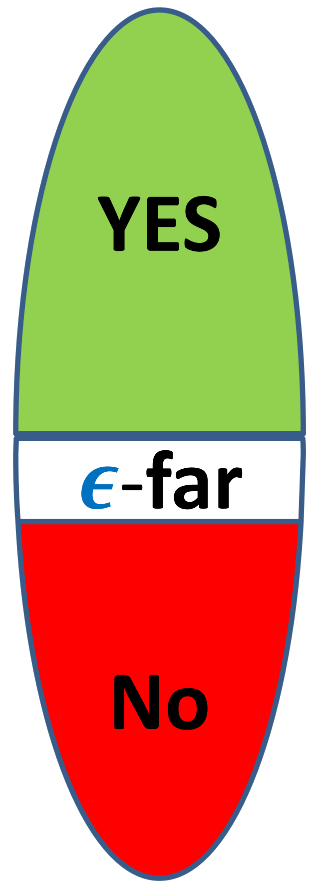
Randomized algorithm



$\Rightarrow$ **Accept with probability** $\geq \frac{2}{3}$

$\Rightarrow$ **Reject with probability** $\geq \frac{2}{3}$

Property tester



$\Rightarrow$ **Accept with probability** $\geq \frac{2}{3}$

$\Rightarrow$ **Don't care**

$\Rightarrow$ **Reject with probability** $\geq \frac{2}{3}$

ϵ**-far :** $\geq \epsilon$ fraction has to be changed to become **YES**

Property Testing

[Goldreich, Goldwasser, Ron, Rubinfeld, Sudan]

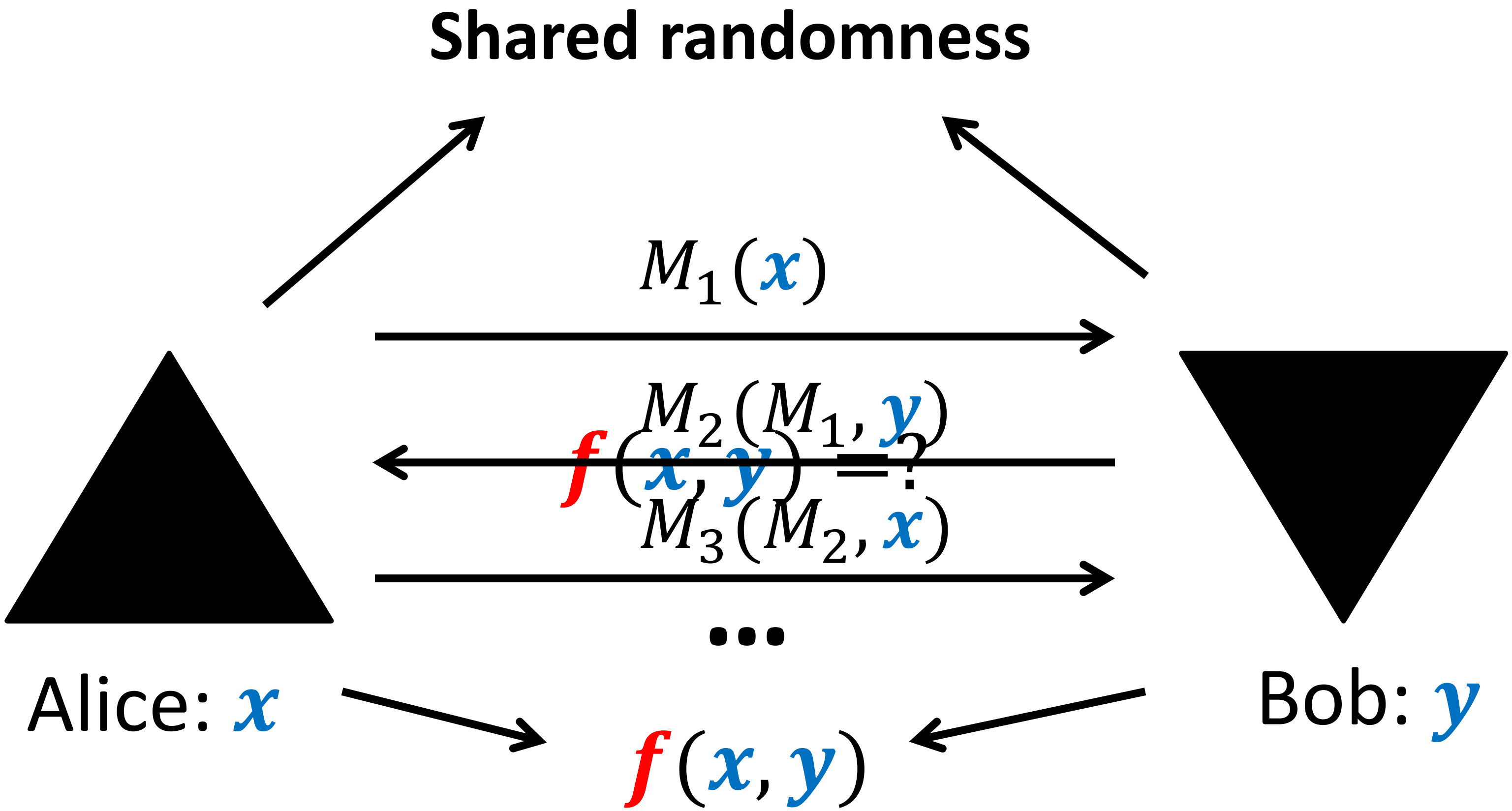
Property $\boldsymbol{P}$ = set of **YES** instances

Query complexity of testing $\boldsymbol{P}$:

- $\boldsymbol{Q}_{\epsilon}(\boldsymbol{P})$ = Adaptive queries
- $\boldsymbol{Q}_{\epsilon}^{na}(\boldsymbol{P})$ = Non-adaptive (all queries at once)
- $\boldsymbol{Q}_{\epsilon}^{r}(\boldsymbol{P})$ = Queries in r rounds ($\boldsymbol{Q}_{\epsilon}^{na}(\boldsymbol{P}) = \boldsymbol{Q}_{\epsilon}^{1}(\boldsymbol{P})$)

For error $1 - \boldsymbol{\delta}$: $\boldsymbol{Q}_{\epsilon,\boldsymbol{\delta}}^{r}(\boldsymbol{P}) = O(\log 1/\boldsymbol{\delta})\boldsymbol{Q}_{\epsilon}^{r}(\boldsymbol{P})$

Communication Complexity [Yao'79]



- $R(f)$ = min. communication (error $1/3$)
- $R^k(f) =$ min. k-round communication (error $1/3$)

$\boldsymbol{k}/2$-disjointness $\Rightarrow$ $\boldsymbol{k}$-linearity

[Blais, Brody,Matulef'11]

- $\boldsymbol{k}$-linear function: $\{0,1\}^n \rightarrow \{0,1\}$

$$\bigoplus_{i \in \boldsymbol{S}} x_i = x_{i_1} \oplus x_{i_2} \oplus \cdots \oplus x_{i_{\boldsymbol{k}}}$$

where $|\boldsymbol{S}| = \boldsymbol{k}$

- $\boldsymbol{k}/2$-Disjointness: $\boldsymbol{S}, \boldsymbol{T} \subseteq [n]$, $|\boldsymbol{S}| = |\boldsymbol{T}| = \frac{\boldsymbol{k}}{2}$

$$\boldsymbol{f}(\boldsymbol{S}, \boldsymbol{T}) = 1, \text{ iff } |\boldsymbol{S} \cap \boldsymbol{T}| = 0.$$

$\boldsymbol{f}$: $|\boldsymbol{S} \cap \boldsymbol{T}| = 0$?

Alice:

$\boldsymbol{S} \subseteq [\boldsymbol{n}], |\boldsymbol{S}| = \boldsymbol{k}/\mathbf{2}$

Bob:

$\mathbf{T} \subseteq [\boldsymbol{n}], |\boldsymbol{T}| = \boldsymbol{k}/\mathbf{2}$

$\boldsymbol{k}$/2-disjointness ⇒ $\boldsymbol{k}$-linearity

[Blais, Brody,Matulef’11]

$$\chi = \chi_S \oplus \chi_T$$

$\boldsymbol{S} \subseteq [\boldsymbol{n}], |\boldsymbol{S}| = \boldsymbol{k}/\mathbf{2}$

$\chi_S = \bigoplus_{i \in S} x_i$

$\mathbf{T} \subseteq [\boldsymbol{n}], |\boldsymbol{T}| = \boldsymbol{k}/\mathbf{2}$

$\chi_T = \bigoplus_{i \in T} x_i$

- $S \cap T = \emptyset \Rightarrow \chi$ is $\boldsymbol{k}$-linear
- $S \cap T \neq \emptyset \Rightarrow \chi$ is $(< \boldsymbol{k})$-linear, ½-far from $\boldsymbol{k}$-linear

- Test χ for $\boldsymbol{k}$-linearity using shared randomness
- To evaluate $\chi(x)$ exchange $\chi_S(x)$ and $\chi_T(x)$ (2 bits)
- $\boldsymbol{R}\left(\frac{\boldsymbol{k}}{2}\text{–Disjointness}\right) \leq 2 \cdot \boldsymbol{Q}_{\frac{1}{2}}(\boldsymbol{k}\text{–Linearity})$

$\boldsymbol{k}$-Disjointness

- $\boldsymbol{R}(\boldsymbol{k}\text{–Disjointness}) = \Theta(\boldsymbol{k})$ [Razborov, Hastad-Wigderson]
- $\boldsymbol{R}^{\mathbf{1}}(\boldsymbol{k}\text{–Disjointness}) = \Theta(\boldsymbol{k} \log \boldsymbol{k})$

[Folklore + Dasgupta, Kumar, Sivakumar; Buhrman'12, Garcia-Soriano, Matsliah, De Wolf'12]

- $\boldsymbol{R}^{\boldsymbol{r}}(\boldsymbol{k}\text{–Disjointness}) = \Theta(\boldsymbol{k} \text{ ilog}^{r} \boldsymbol{k})$,

where $ilog^{r} k = \underbrace{\log \log \ldots \log}_{r \text{ times}} k$ [Saglam, Tardos'13]

$\Omega(\boldsymbol{k} \text{ ilog}^{r} \boldsymbol{k}) = \boldsymbol{Q}^{\boldsymbol{r}}{}_{1/2}(\boldsymbol{k}\text{–Linearity}) = O(\boldsymbol{k} \log \boldsymbol{k})$

- $\boldsymbol{R}(\boldsymbol{k}\text{–Disjointness}) = \alpha \boldsymbol{k} + o(\boldsymbol{k})$ [Braverman, Garg, Pankratov, Weinstein'13]
- $\boldsymbol{R}^{\boldsymbol{r}}(\boldsymbol{k}\text{-Intersection}) = \Omega(\boldsymbol{k}\ ilog^{r} \boldsymbol{k}), O\left(\boldsymbol{k}\ ilog^{\boldsymbol{\beta} \boldsymbol{r}} \boldsymbol{k}\right)$

[Brody, Chakrabarti, Kondapally, Woodruff, Y.]

Communication Direct Sums

"Solving **m** copies of a communication problem requires **m** times more communication":

$$R^r(f^m) = \Omega(m) R^r(f)$$

- For arbitrary f [… Braverman, Rao 10; Barak Braverman, Chen, Rao 11, ….]
- In general, can't go beyond

$EQ_m(x, y) = 1$ iff $x = y$, where $x, y \in \{0,1\}^m$

$$R(EQ_m) = O(1)$$

$$R\left(EQ_{,m}^m\right) = O(m)$$

Specialized Communication Direct Sums

Information cost $\leq$ Communication complexity

- $\mathrm{R}(\mathrm{Disjointness}) = \Omega(n)$ [Bar Yossef, Jayram, Kumar,Sivakumar'01]

  $$\text{Disjointness}(x, y) = \wedge_i(\neg x_i \vee \neg y_i)$$

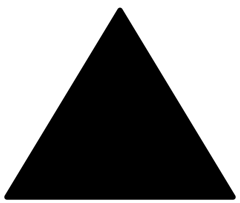
- Stronger direct sum for Equality-type problems (a.k.a. "union bound is optimal") [Molinaro, Woodruff, Y.'13]

  $$\boldsymbol{R}^1(EQ^{\boldsymbol{m}}) = \Omega(\boldsymbol{m} \log \boldsymbol{m})\boldsymbol{R}^1(EQ)$$

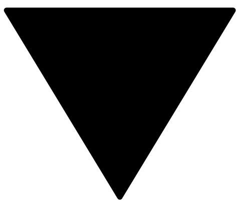
- Bounds for $\boldsymbol{R}^{\boldsymbol{r}}(EQ^{\boldsymbol{m}})$, $\boldsymbol{R}^r(\boldsymbol{k}$-Set Intersection) via Information Theory [Brody, Chakrabarty, Kondapally, Woodruff, Y.'13]

Direct Sums in Property Testing [Woodruff, Y.]

- Testing linearity: $\boldsymbol{f}$ is linear if $\boldsymbol{f} = \bigoplus_{i \in \boldsymbol{S}} x_i$
- Equality: $\boldsymbol{S}, \boldsymbol{T} \subseteq [n]$ decide whether $\boldsymbol{S} = \boldsymbol{T}$



$$\chi = \chi_{\boldsymbol{S}} \oplus \chi_{\boldsymbol{T}}$$



$\boldsymbol{S} \subseteq [\boldsymbol{n}]$

$\chi_{\boldsymbol{S}} = \bigoplus_{i \in \boldsymbol{S}} (x_{2i-1} \wedge x_{2i})$

$\mathbf{T} \subseteq [\boldsymbol{n}]$

$\chi_{\boldsymbol{T}} = \bigoplus_{i \in \boldsymbol{T}} (x_{2i-1} \wedge x_{2i})$

- $S = T \Rightarrow \chi$ is linear
- $S \neq T \Rightarrow \chi$ is ¼ -far from linear

Direct Sums in Property Testing [Woodruff, Y.]

- $R_{\boldsymbol{\delta}}(\mathrm{E}Q) = \Omega(\log 1/\boldsymbol{\delta}) \Rightarrow Q_{1/4}(Lin) = \Omega(\log\frac{1}{\boldsymbol{\delta}})$ (matching [Blum, Luby, Rubinfeld])

- **Strong Direct Sum for Equality** [MWY'13] $\Rightarrow$
  Strong Direct Sum for Testing Linearity
  $\boldsymbol{Q}^1(\mathrm{Lin}^{\boldsymbol{m}}) \geq$
  $\boldsymbol{R}^1(EQ_{\boldsymbol{m}}^{\boldsymbol{m}}) =$
  $\Omega(\boldsymbol{m}\log\boldsymbol{m}) =$
  $\Omega(\boldsymbol{m}\log\boldsymbol{m})\boldsymbol{Q}^1(\mathrm{Lin})$

Property Testing Direct Sums [Goldreich'13]

- Direct Sum [Woodruff, Y.]:

  Solve $P^{\boldsymbol{m}}$ with probability $\geq \frac{2}{3}$

- Direct $\boldsymbol{m}$-Sum[Goldreich'13]:

  Solve $P^{\boldsymbol{m}}$ with probability $\geq \frac{2}{3}$ per instance

- Direct $\boldsymbol{m}$-Product[Goldreich'13]:

  All instances are in P $\boldsymbol{vs}$.

  $\exists$ instance $\boldsymbol{\epsilon}$–far from P

[Goldreich '13]

For all properties P:

- Direct **m**-Sum (solve all w.p. 2/3 per instance)
  - Adaptive:

$$\boldsymbol{Q}(\boldsymbol{DS}^{\boldsymbol{m}}_{\epsilon}(P)) = \Theta(\boldsymbol{m}\,\boldsymbol{Q}_{\epsilon}(P))$$

  - Non-adaptive:

$$\boldsymbol{Q}^{\mathbf{1}}(\boldsymbol{DS}^{\boldsymbol{m}}_{\epsilon}(P)) = \Theta(\boldsymbol{m}\,\boldsymbol{Q}^{1}_{\epsilon}(P))$$

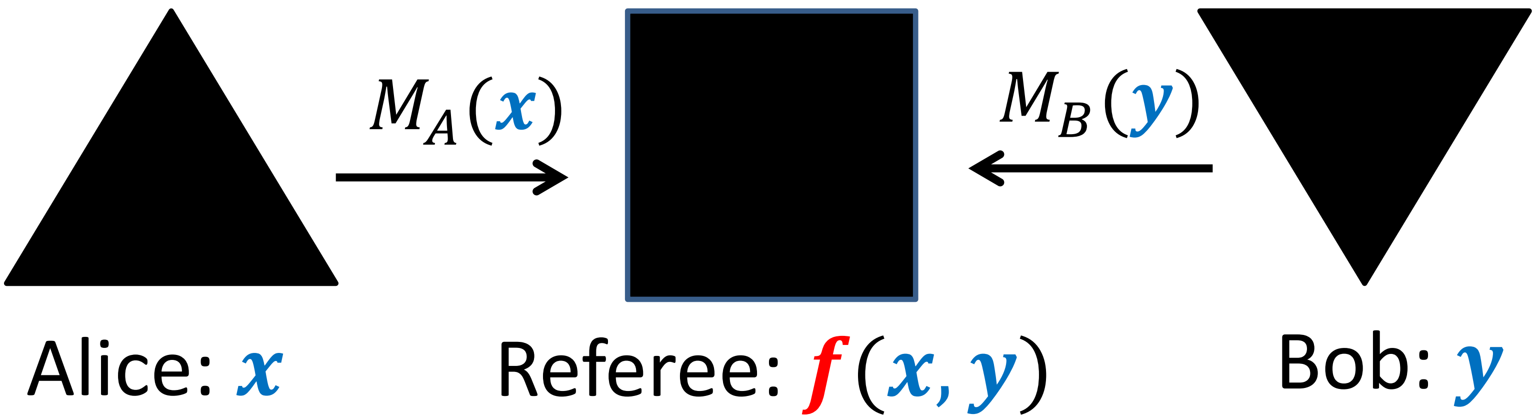
- Direct $\boldsymbol{m}$-Product (All in P $\boldsymbol{vs}.$ $\exists$ ϵ-far instance?)
  - Adaptive:

$$\boldsymbol{DP}^{\boldsymbol{m}}_{\epsilon}(P) = \Theta(\boldsymbol{m}\,\boldsymbol{Q}_{\epsilon}(P))$$

  - Non-adaptive:

$$\Omega(\boldsymbol{m}\,\boldsymbol{Q}^{1}_{\epsilon}(P)) = \boldsymbol{Q}^{\mathbf{1}}(\boldsymbol{DP}^{\boldsymbol{m}}_{\epsilon}(P)) = O(\boldsymbol{m}\log\boldsymbol{m}\,\boldsymbol{Q}^{1}_{\epsilon}(P))$$

Reduction from Simultaneous Communication [Woodruff]



- $S(\boldsymbol{f})$ = min. simultaneous complexity of $\boldsymbol{f}$
- $\boldsymbol{R^{1,A\to B}(f), R^{1,B\to A}(f) \leq S(f)}$
- GAF: $\{0,1\}^{2n+2\log n} \to \{0,1\}$ [Babai, Kimmel, Lokam]

GAF$(a, x, b, y) = a_{x\oplus y}$ if $a = b$, 0 otherwise

$R^{1,A\to B}(GAF) = O(\log\ n)$, but S(GAF) = $\Omega(\sqrt{n})$

Property testing lower bounds via CC

- Monotonicity, Juntas, Low Fourier degree, Small Decision Trees [Blais, Brody, Matulef’11]
- Small-width OBDD properties [Brody, Matulef, Wu’11]
- Lipschitz property [Jha, Raskhodnikova’11]
- Codes [Goldreich’13, Gur, Rothblum’13]
- Number of relevant variables [Ron, Tsur’13]

All functions are over Boolean hypercube

Functions $[\boldsymbol{m}]^{\boldsymbol{n}} \to \mathbb{R}$ [Blais, Raskhodnikova, Y.]

$M_{\boldsymbol{m},\boldsymbol{n}}$ = monotone functions over $[\boldsymbol{m}]^{\boldsymbol{n}}$

$$\boldsymbol{Q}^1(M_{\boldsymbol{m},\boldsymbol{n}}) = \Omega(\boldsymbol{n} \log \boldsymbol{m})$$

Previous for monotonicity on the line ($\boldsymbol{n} = 1$):

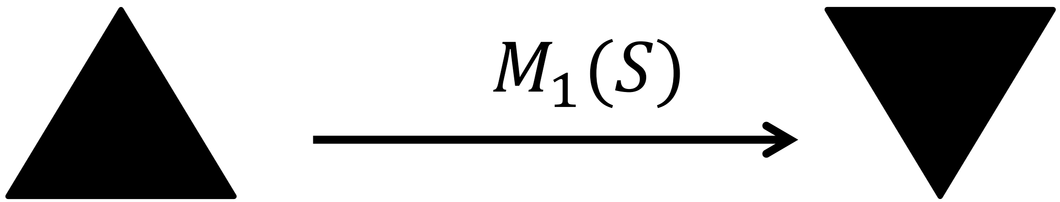
- $\boldsymbol{Q}^1(M_{\boldsymbol{m},\boldsymbol{1}}) = \Theta(\log \boldsymbol{m})$ [Ergun, Kannan, Kumar, Rubinfeld, Viswanathan'00]
- $\boldsymbol{Q}(M_{\boldsymbol{m},\boldsymbol{1}}) = \Omega(\log \boldsymbol{m})$ [Fischer'04]

Functions $[\boldsymbol{m}]^{\boldsymbol{n}} \to \mathbb{R}$ [Blais, Raskhodnikova, Y.]

- **Thm.** Any non-adaptive tester for monotonicity of $f: [\boldsymbol{m}] \to [r]$ has complexity
$$\Omega(\min(\log \mathbf{m}, \log \mathbf{r}))$$
- **Proof.**
  - Reduction from Augmented Index
  - Basis of Walsh functions

Functions $[\boldsymbol{m}]^{\boldsymbol{n}} \to \mathbb{R}$ [Blais, Raskhodnikova, Y.]

- **Augmented Index**: **S**, $(\boldsymbol{i}, \boldsymbol{S} \cap [\boldsymbol{i} - 1])$



$\boldsymbol{S} \subseteq [\boldsymbol{m}]$ $\qquad$ $\boldsymbol{i} \in [\boldsymbol{m}], \boldsymbol{S} \cap [\boldsymbol{i} - 1]$

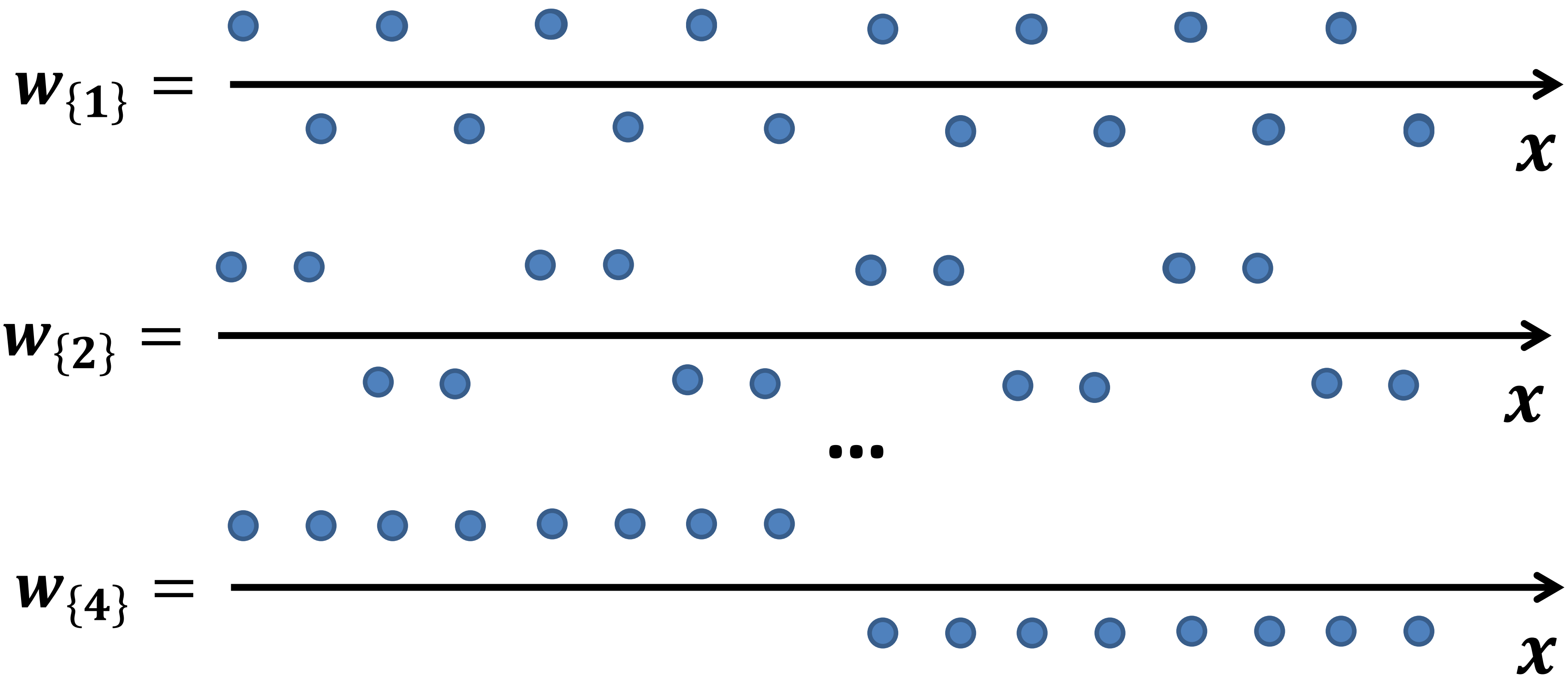
- $\boldsymbol{R}^1[\text{Augmented Index}] = \Omega(\boldsymbol{m})$ [Miltersen, Nisan, Safra, Wigderson, 98]

Functions $[\boldsymbol{m}]^{\boldsymbol{n}} \to \mathbb{R}$ [Blais, Raskhodnikova, Y.]

Walsh functions: For $S \subseteq [m]$, $w_S\colon [2^m] \to \{-1,1\}$:

$$w_S(x) = \prod_{i \in S} (-1)^{x_i},$$

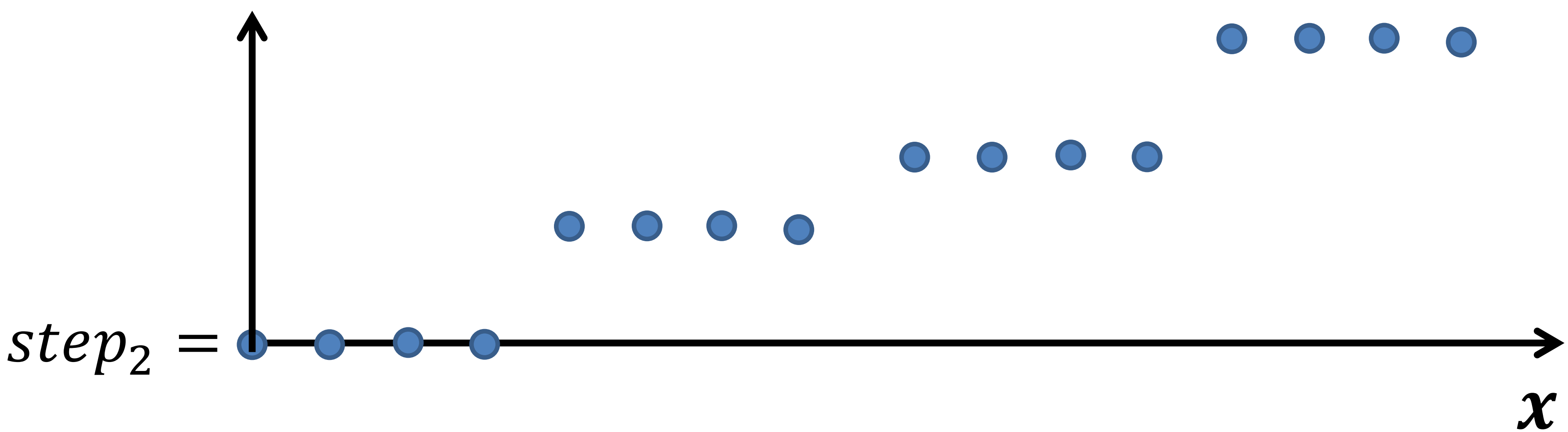
where x_i is the i-th bit of x.



Functions $[\boldsymbol{m}]^{\boldsymbol{n}} \to \mathbb{R}$ [Blais, Raskhodnikova, Y.]

Step functions: For $i \in [m], step_i \colon [2^m] \to [2^{m-i}]$:

$$step_i(x) = \lceil x/2^i \rceil$$



Functions $[\boldsymbol{m}]^{\boldsymbol{n}} \to \mathbb{R}$ [Blais, Raskhodnikova, Y.]

- **Augmented Index** $\Rightarrow$ Monotonicity Testing

$$\chi = 2\, step_{\boldsymbol{i}} + \boldsymbol{w}_{\boldsymbol{S} \cap [\boldsymbol{i}-1,\ldots,\boldsymbol{m}]}$$

$\boldsymbol{S} \subseteq [\boldsymbol{m}]$

$\boldsymbol{i} \in [\boldsymbol{m}], \boldsymbol{S} \cap [\boldsymbol{i} - 1]$

- $i \notin S \Rightarrow \chi$ is monotone
- $i \in S \Rightarrow \chi$ is ¼ -far from monotone
- Thus, $\boldsymbol{Q}^{\mathbf{1}}\left(M_{\boldsymbol{m},\mathbf{1}}\right) = \Omega(\log \boldsymbol{m})$

Functions $[\boldsymbol{m}]^{\boldsymbol{n}} \to \mathbb{R}$ [Blais, Raskhodnikova, Y.]

- $M_{\boldsymbol{m},\boldsymbol{n}}$ = monotone functions over $[\boldsymbol{m}]^{\boldsymbol{n}}$

$$\boldsymbol{Q}^1(M_{\boldsymbol{m},\boldsymbol{n}}) = \Omega(\boldsymbol{n} \log \boldsymbol{m})$$

- $L_{\boldsymbol{m},\boldsymbol{n}} =$ c-Lipschitz functions over $[\boldsymbol{m}]^{\boldsymbol{n}}$
- $C^S_{\boldsymbol{m},\boldsymbol{n}}$ = separately convex functions over $[\boldsymbol{m}]^{\boldsymbol{n}}$
- $C_{\boldsymbol{m},\boldsymbol{n}}$ = convex functions over $[\boldsymbol{m}]^{\boldsymbol{n}}$

Thm. [BRY] For all these properties $\boldsymbol{Q}^1 = \Omega(\boldsymbol{n} \log \boldsymbol{m})$

These bounds are optimal for $M_{\boldsymbol{m},\boldsymbol{n}}$ and $L_{\boldsymbol{m},\boldsymbol{n}}$ [Chakrabarty, Seshadhri, '13]

Thank you!