Steiner Transitive-Closure Spanners of Low-Dimensional Posets

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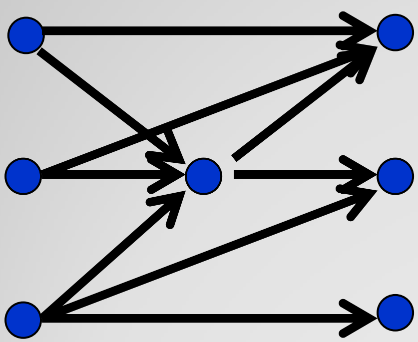
Graph Spanners

k-spanner (stretch k): Graph G(V,E) → H(V, $E_H \subseteq E$), if for all pairs of vertices (u, v) in G:

$$distance_H(u,v) \leq k \cdot \; distance_G(u,v)$$



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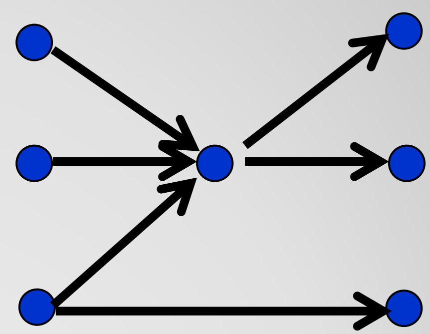
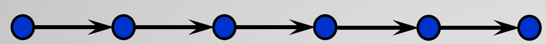


H: 2-spanner of *G*

[Awerbuch '85, Peleg-Schäffer '89]

Transitive-Closure Spanners

Transitive closure TC(G) has an edge from u to v iff G has a path from u to v



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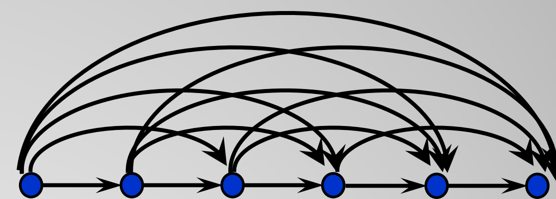
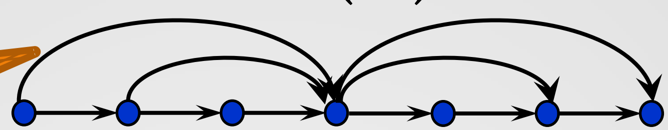


TC(G)

H is a ***k*-TC-spanner** of G if H is a subgraph of TC(G) for which $distance_H(u,v) \leq k$ iff G has a path from u to v

H is a ***k*-TC-spanner** of G if H is a k-spanner of TC(G)



2-TC-spanner of G

Shortcut edge consistent with ordering

[Bhattacharyya, Grigorescu, Jung, Raskhodnikova, Woodruff, SODA'09], generalizing [Yao 82; Chazelle 87; Alon, Schieber 87, …]

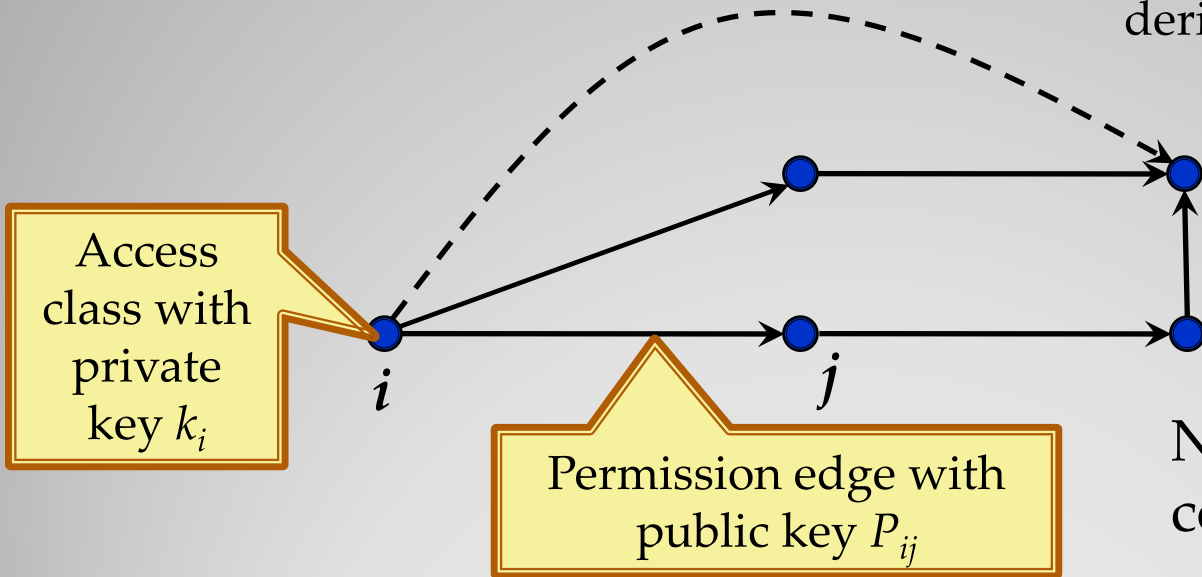
Applications of TC-Spanners

- Data structures for storing partial products [Yao, ’82; Chazelle ’87, Alon, Schieber, 88]
- Constructions of unbounded fan-in circuits [Chandra, Fortune, Lipton ICALP, STOC‘83]
- Property testers for monotonicity and Lipschitzness [Dodis et al. ‘99,BGJRW’09; Jha, Raskhodnikova, FOCS ‘11]
- Lower bounds for reconstructors for monotonicity and Lipshitzness [BGJJRW’10, JR’11]
- Efficient key management in access hierarchies

Follow references in [Raskhodnikova ’10 (survey)]

Application to Access Control

Sparse TC-spanner ⇒ Access hierarchy with low storage and low key derivation times

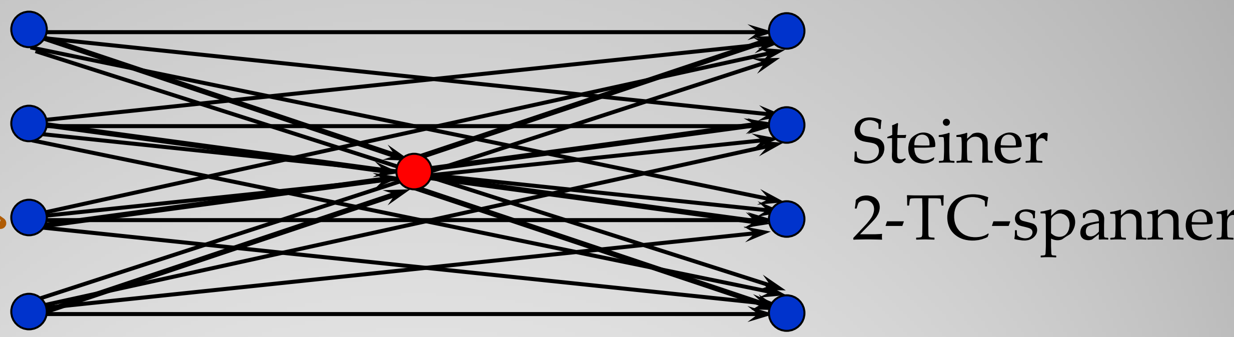


Need k_i to efficiently compute k_j from P_{ij}

To speed up key derivation time, add shortcut edges consistent with permission edges

[Attalah, Frikken, Blanton '05; Attalah,Blanton,Frikken '06, De Santis,Ferrara,Massuci '07, Attalah,Blanton,Fazio,Frikken '09]

Steiner TC-Spanners



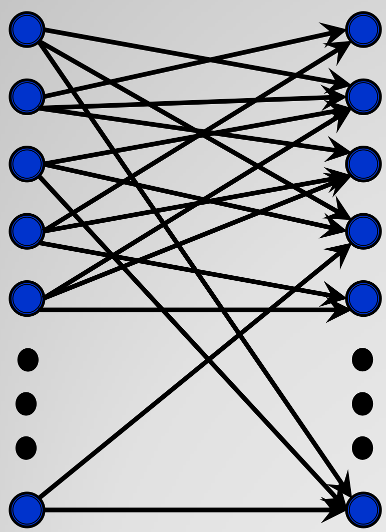
H is a **Steiner *k*-TC-spanner** of *G* if

- vertices(G) $\subseteq$ vertices(H)
- $distance_H(u,v) \leq k$ if *G* has a path from *u* to *v*

  ∞ otherwise

Bounds for Steiner TC-Spanners

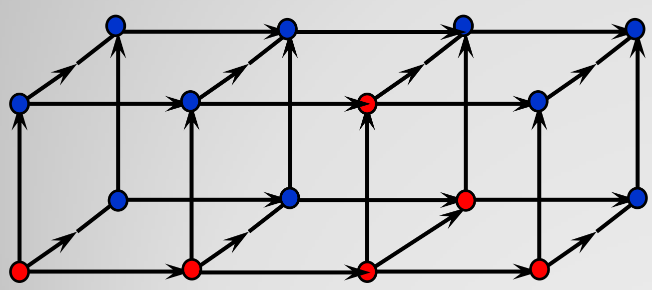
- For some graphs Steiner TC-spanners are still large:



FACT: For a random directed bipartite graph of density ½, any Steiner 2-TC-spanner requires $\tilde{\Omega}(n^2)$ edges.

Low-Dimensional Posets

- [ABFF 09] access hierarchies are **low-dimensional posets**
- Poset ≡ DAG (edge $(u, v) \equiv$ u $\preccurlyeq v$)
- Poset G has dimension d if G can be <u>embedded</u> into a <u>hypergrid</u> of dimension d and d is minimum.



$e: G \rightarrow G'$ is a **poset embedding** if for all $x, y \in G$, $x \preccurlyeq_G y$ iff $e(x) \preccurlyeq_{G'} e(y)$.

Hypergrid $[m]^d$ has ordering $(x_1, \ldots, x_d) \preccurlyeq (y_1, \ldots, y_d)$ iff $x_i \leq y_i$ for all i

Our Main Results

Stretch k	Upper Bound	Lower Bound
2	$O(n \log^d n)$	$\boxed{\Omega\left(n\left(\frac{\alpha \log n}{d}\right)^d\right)}$ where α is a constant
≥ 3	$O(n \log^{d-1} n \log\log n)$ for constant d [DFM 07]	$\Omega(n \log^{\lceil (d-1)/k \rceil} n)$ for constant d

d is the poset dimension

$k = 3$. Nothing better known for larger k.

Facts about Poset Dimension

- Hypergrid $[m]^d$ has dimension exactly d. [Dushnik-Miller 40]
- Finding a poset dimension is NP-hard. [Yannakakis 82]
  - Assumption (as in [ABFF 09]): poset embedding into a hypergrid of minimal dimension is explicitly given.

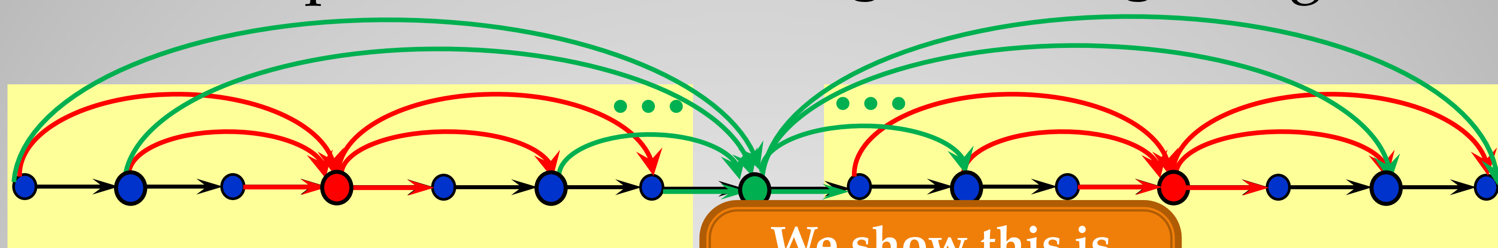
Lower Bound for $k = 2$

- We prove lower bound for size of Steiner 2-TC-spanner for the hypergrid $[m]^d$.
  - Steiner points can be embedded
  - Minimal size of Steiner 2-TC-spanner = Minimal size of (non-Steiner) 2-TC-spanner
  - Lower bound for size of 2-TC-spanner

2-TC-Spanner for $[m]^d$

- $d = 1$ (so, $n = m$)

  2-TC-spanner with $\leq m \log m = n \log n$ edges



We show this is tight upto α^d for a constant α.

- $d > 1$ (so, $n = m^d$)

  2-TC-spanner with $\leq (m \log m)^d = n \left(\frac{\log n}{d}\right)^d$ edges by taking d-wise Cartesian product of 2-TC-spanners for a line.

Lower Bound Strategy

- Write in IP for a minimal 2-TC-spanner.
- OPT $\geq LP = LP_{dual}$
- Integrality gap of the primal is small.

- **Idea:** Construct some feasible solution for the dual => lower bound on OPT.
- OPT $\geq LP = LP_{dual} \geq LB$

IP Formulation

- {0,1}-program for Minimal 2-TC-spanner:

minimize

$$\sum_{u,v:u \preccurlyeq v} x_{uv}$$

subject to:

$$x_{uw} \geq p_{uwv} \qquad \forall u \preccurlyeq w \preccurlyeq v$$

$$x_{wv} \geq p_{uwv} \qquad \forall u \preccurlyeq w \preccurlyeq v$$

$$\sum_{w:u \preccurlyeq w \preccurlyeq v} p_{uwv} \geq 1 \qquad \forall u \preccurlyeq v$$

Dual LP

- Take fractional relaxation of IP and look at its dual:

$$\text{maximize} \qquad \sum_{u,v:u \preccurlyeq v} y_{uv}$$

$$\text{subject to:} \sum_{w:v \preccurlyeq w} q_{uvw} + \sum_{w:w \preccurlyeq u} r_{wuv} \leq 1 \quad \forall u \preccurlyeq v$$

$$y_{uv} \leq q_{uwv} + r_{uwv} \qquad \forall u \preccurlyeq w \preccurlyeq v$$

$$0 \leq y_{uv}, q_{uwv}, r_{uwv} \leq 1 \qquad \forall u \preccurlyeq w \preccurlyeq v$$

Constructing solution to dual LP

- Now we use fact that poset is a hypergrid! For $u \preccurlyeq v$, set $y_{uv} = \frac{1}{V(v-u)}$, where $V(v-u)$ is volume of box with corners u and v.

maximize $\sum_{u,v:u \preccurlyeq v} y_{uv}$

$> (m \ln m)^d$

subject to:

$$\sum_{w:v \preccurlyeq w} q_{uvw} + \sum_{w:w \preccurlyeq u} r_{wuv} \leq 1 \quad \forall u \preccurlyeq v$$

$\leq (4\pi)^d$

$$y_{uv} = q_{uwv} + r_{uwv} \quad \forall u \preccurlyeq w \preccurlyeq v$$

Set $q_{uwv} = y_{uv} \frac{V(w-u)}{V(w-u)+V(v-w)}$, $r_{uwv} = y_{uv} \frac{V(v-w)}{V(w-u)+V(v-w)}$

Constructing solution to dual LP

- Now we use fact that poset is a hypergrid! For $u \preccurlyeq v$, set $y_{uv} = \frac{1}{(4\pi)^d V(v-u)}$, where $V(v-u)$ is volume of box with corners u and v.

maximize $$\sum_{u,v:u \preccurlyeq v} y_{uv}$$

$> (m \ln m)^d / (4\pi)^d$

subject to:

$$\sum_{w:v \preccurlyeq w} q_{uvw} + \sum_{w:w \preccurlyeq u} r_{wuv} \leq 1 \quad \forall u \preccurlyeq v$$

≤ 1

$$y_{uv} = q_{uwv} + r_{uwv} \quad \forall u \preccurlyeq w \preccurlyeq v$$

Set $q_{uwv} = y_{uv} \frac{V(w-u)}{V(w-u)+V(v-w)}$, $r_{uwv} = y_{uv} \frac{V(v-w)}{V(w-u)+V(v-w)}$

Wrap-up

- Upper bound for Steiner 2-TC-spanner
- Lower bound for 2-TC-spanner for a hypergrid.
  - Previous bound: combinatorial, ~10 pages, not tight.
  - Technique: find a feasible solution for the dual LP
- Lower bound of $\Omega(n \log^{\lceil (d-1)/k \rceil} n)$ for $k \geq 3$.
  - Combinatorial
  - Holds for randomly generated posets, not explicit.
- OPEN PROBLEM:
  - Can the LP technique give a better lower bound for $k \geq 3$?

THANKS!