Improved Approximation for the Directed Spanner Problem

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Directed Spanner Problem

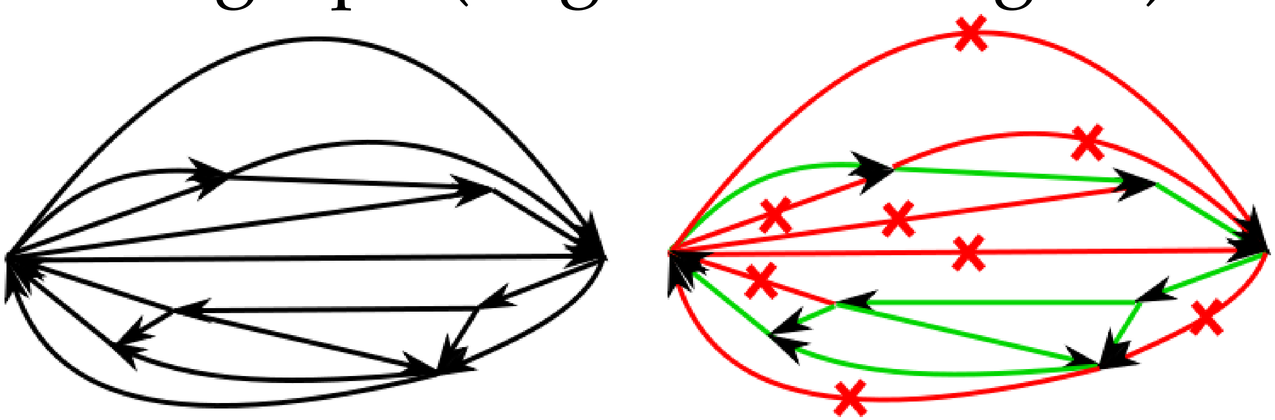
- **k-Spanner** [Awerbuch '85, Peleg, Shäffer '89]

Subset of edges, preserving distances up to a factor k > 1 (stretch k).

- Graph G(V, E) → k-spanner **H(V, $E_H \subseteq E$)**:

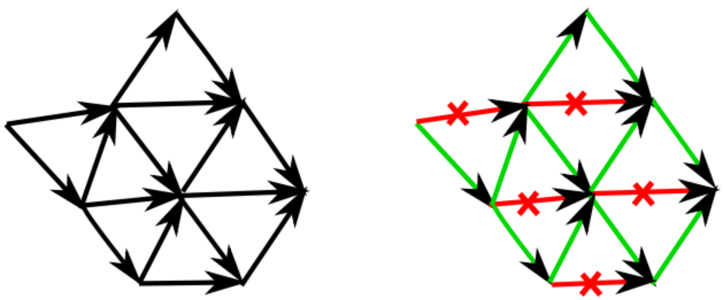
$$\forall u, v \in V \quad dist_H(u, v) \leq k \cdot dist_G(u, v)$$

- **Problem:** Find the sparsest k-spanner of a directed graph (edges have lengths).

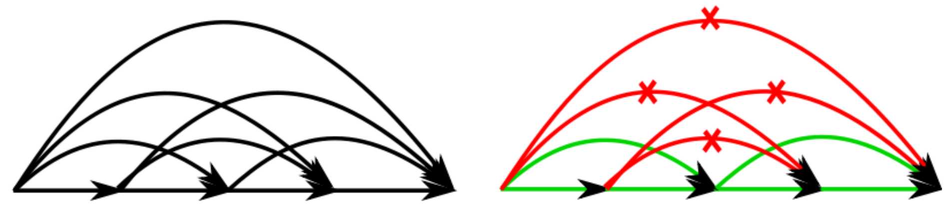


Directed Spanners and Their Friends

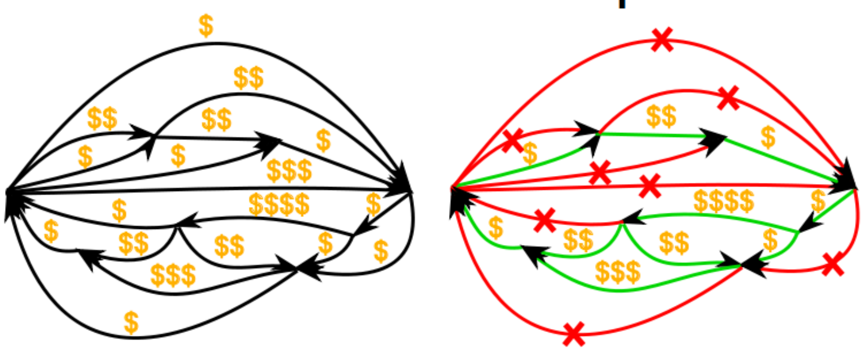
Unit lengths



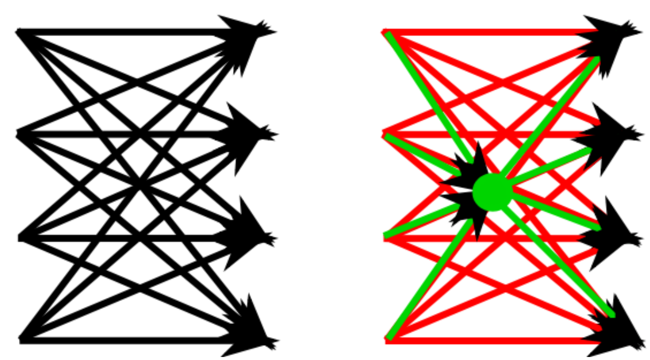
Transitive-closure spanner



Minimum cost spanner



Steiner spanner



Applications of spanners

- First application: simulating synchronized protocols in unsynchronized networks [Peleg, Ullman '89]
- Efficient routing [PU'89, Cowen '01, Thorup, Zwick '01, Roditty, Thorup, Zwick '02 , Cowen, Wagner '04]
- Parallel/Distributed/Streaming approximation algorithms for shortest paths [Cohen '98, Cohen '00, Elkin'01, Feigenbaum, Kannan, McGregor, Suri, Zhang '08]
- Algorithms for approximate distance oracles [Thorup, Zwick '01, Baswana, Sen '06]

Applications of directed spanners

- Access control hierarchies
  - Previous work: [Atallah, Frikken, Blanton, CCCS '05; De Santis, Ferrara, Masucci, MFCS'07]
  - Solution: [Bhattacharyya, Grigorescu, Jung, Raskhodnikova, Woodruff, SODA'09]
  - Steiner spanners for access control: [Berman, Bhattacharyya, Grigorescu, Raskhodnikova, Woodruff, Y' ICALP'11 (more on Friday)]
- Property testing and property reconstruction [BGJRW'09; Raskhodnikova '10 (survey)]

Plan

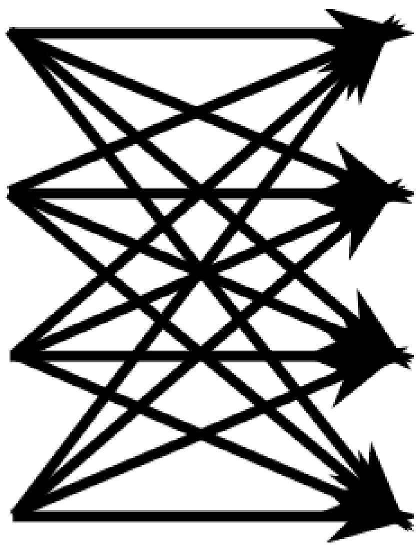
- Undirected vs Directed
- Previous work
- Framework = Sampling + LP
- Sampling
- LP + Randomized rounding
  - Directed Spanner
  - Unit-length 3-spanner
  - Directed Steiner Forest

Undirected vs Directed

- Every undirected graph has a (2t-1)-spanner with $\leq n^{1+1/t}$ edges. [Althofer, Das, Dobkin, Joseph, Soares '93]
  - Simple greedy + girth argument
  - $n^{\frac{1}{t}}$ −approximation
- Time/space-efficient constructions of undirected approximate distance oracles [Thorup, Zwick, STOC '01]

Undirected vs Directed

- For some directed graphs $\Omega(n^2)$ edges needed for a k-spanner:



- No space-efficient directed distance oracles: some graphs require $\Omega(n^2)$ space. [TZ ‘01]

Unit-Length Directed k-Spanner

- O(n)-approximation: trivial (whole graph)

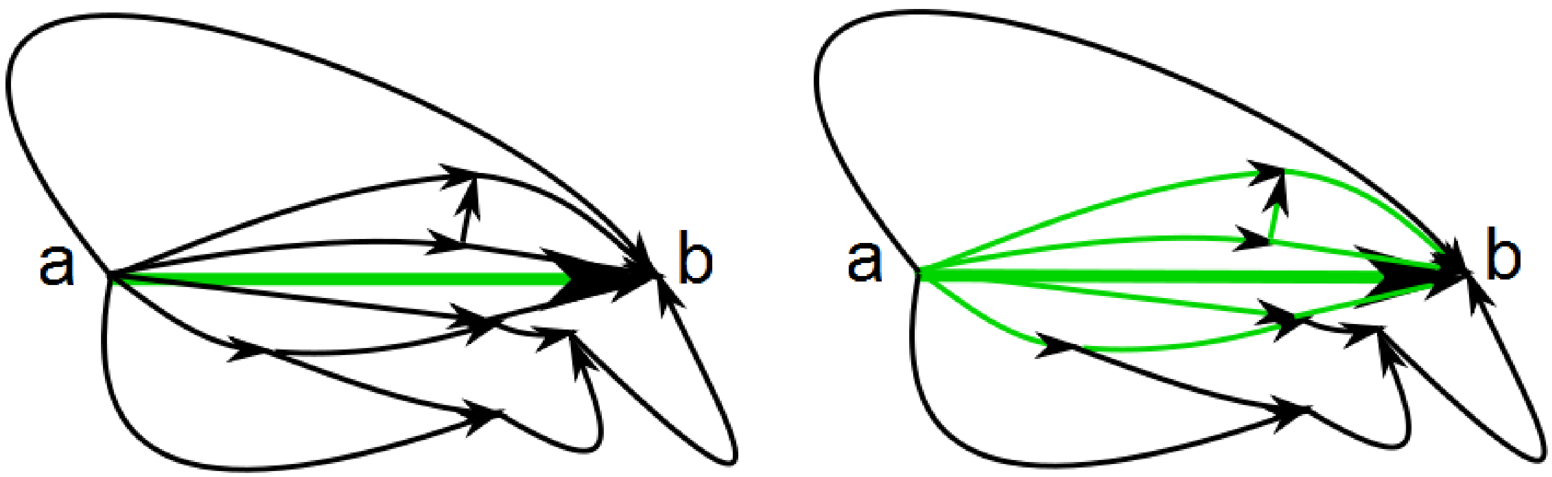
Stretch	$k = 2$	$k = 3$	$k \geq 4$
Previous work	$O(\log n)$ [KP94]	$\tilde{O}(n^{2/3})$ [EP00] $\tilde{O}(n^{2/3})$ [BGJRW09] $\tilde{O}(\sqrt{n})$ [BRR10] $\tilde{O}(\sqrt{n})$ [DK11]	$\tilde{O}(n^{1-\frac{1}{k}})$ [BGJRW09] $\tilde{O}(n^{1-\frac{1}{\lceil k/2 \rceil}})$ [BRR10] $\tilde{O}(n^{2/3})$[DK11]
Our work		$\tilde{O}(n^{1/3})$ + *undirected*!	$\tilde{O}(\sqrt{n})$
Integrality gap	$\Omega(\log n)$ [DK11]	$\Omega(\frac{1}{k} n^{1/3-\epsilon})$ [DK11]	
Hardness	$\Omega(\log n)$ NP-hard [K01]	$2^{\log^{1-\epsilon} n}$ quasi-NP-hard [EP00]	

Overview of the algorithm

- Paths of stretch k for all edges => paths of stretch k for all pairs of vertices
- Classify edges: **thick** and thin
- Take union of spanners for them
  - **Thick** edges: Sampling
  - Thin edges: LP + randomized rounding
- Choose **thickness** parameter to balance approximation

Local Graph

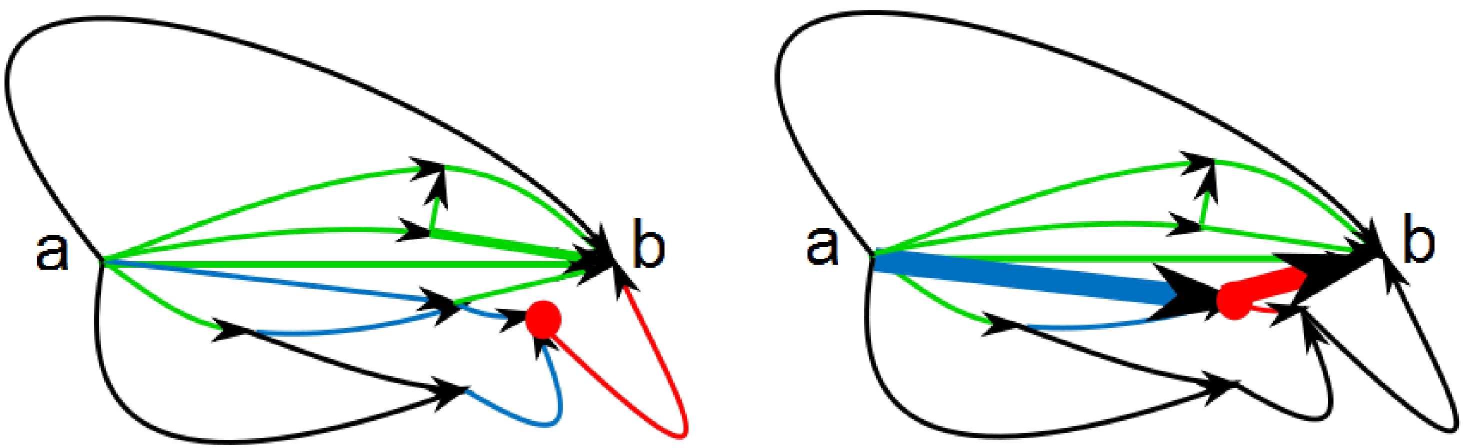
- Local graph for an edge (a,b): Induced by vertices on paths of stretch ≤ k from a to b



- Paths of stretch k only use edges in local graphs
- **Thick** edges: ≥ $\sqrt{n}$ vertices in their local graph. Otherwise thin.

Sampling [BGJRW'09, FKN09, DK11]

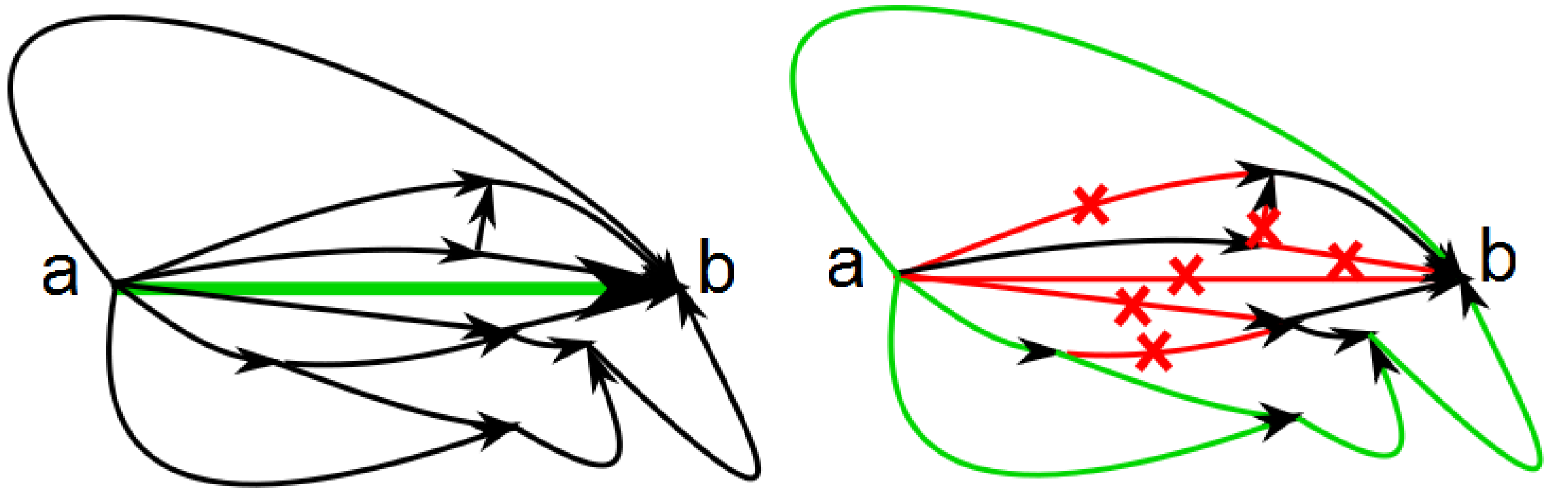
- Pick $\sqrt{n}\ln n$ seed vertices at random
- Add in- and out- shortest path trees for each



- Handles all **thick** edges ($\geq \sqrt{n}$ vertices in their local graph) w.h.p.
- # of edges $\leq 2(n-1)\sqrt{n}\ln n \leq OPT \cdot \tilde{O}(\sqrt{n})$.

Key Idea: Antispanners

- Antispanner – subset of edges, which destroys all paths from **a** to **b** of stretch at most k.



- Spanner <=> hit all antispanners
- Enough to hit all minimal antispanners for all thin edges
- Minimal antispanners can be found efficiently

Linear Program (dual to [DK'11])

Hitting-set LP: $\sum_{e \in E} x_e \to min$

$$\sum_{e \in \mathbf{A}} x_e \geq 1$$

for all minimal antispanners **A** for all thin edges.

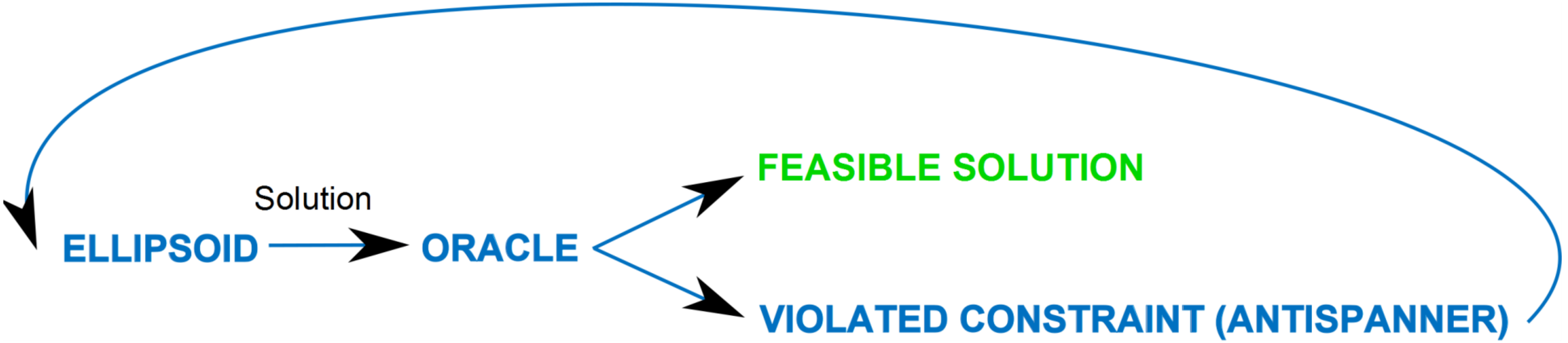
- # of minimal antispanners may be exponential in $\sqrt{n}$ => Ellipsoid + Separation oracle
- Good news: $\leq \sqrt{n}^{\sqrt{n}} = e^{\frac{1}{2}\sqrt{n}\ln n}$ minimal antispanners for a fixed thin edge
- Assume, that we guessed the size of the sparsest k-spanner OPT (at most n^2 values)

Oracle

Hitting-set LP: $\sum_{e \in E} x_e \leq OPT$

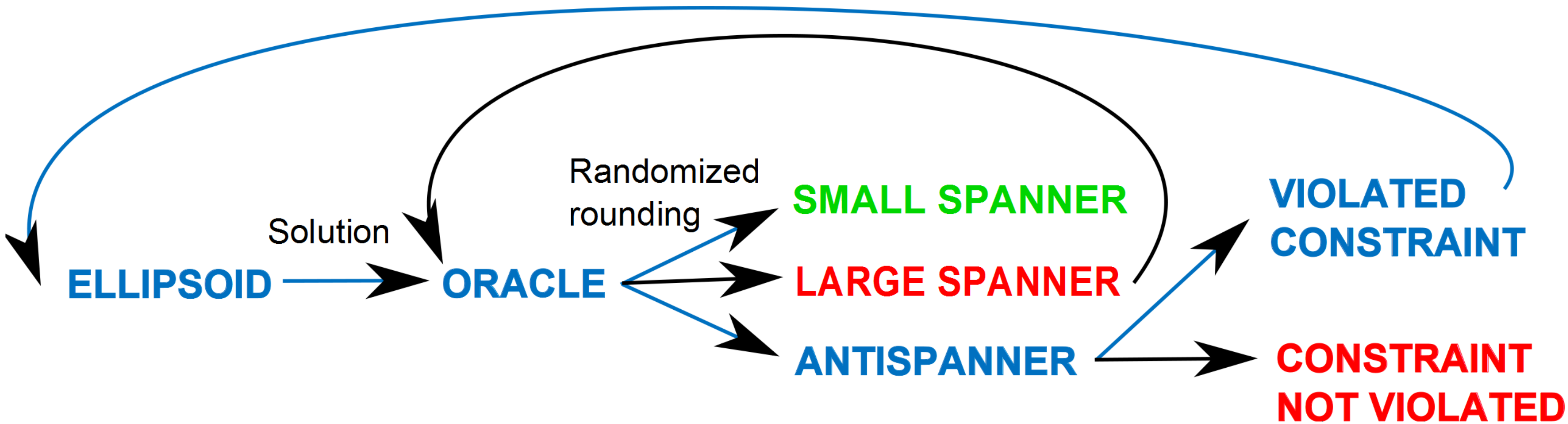
$$\sum_{e \in \mathbf{A}} x_e \geq 1$$

for all minimal antispanners **A** for all thin edges.



- We use a randomized oracle => in both cases oracle can fail with some probability.

Randomized Oracle = Rounding



- Rounding: Take **e** w.p. $p_e = \min\left(\sqrt{n} \ln n \cdot x_e, 1\right)$
- **SMALL SPANNER**: We have a spanner of size $\leq \sum_e x_e \cdot \tilde{O}(\sqrt{n}) \leq OPT \cdot \tilde{O}(\sqrt{n})$ w.h.p.
- Pr[**LARGE SPANNER** **or** **CONSTRAINT NOT VIOLATED**] $\leq e^{-\Omega(\sqrt{n})}$

Unit-length 3-spanner

- $\tilde{O}(n^{1/3})$-approximation algorithm
- Sampling: $\tilde{O}(n^{1/3})$ times
- Dual LP + Different randomized rounding (simplified version of [DK'11])
- For each vertex $u \in V$: sample a real $r_u \in [0,1]$
- Take all edges (u, v):

$$\min(r_u, r_v) \le \tilde{O}(n^{1/3}) x_{(u,v)}$$

- Feasible solution => 3-spanner w.h.p.

Conclusion

- Sampling + LP with randomized rounding
- Improvement for **Directed Steiner Forest**:
  - Cheapest set of edges, connecting pairs (s_i, t_i)
  - Previous: Sampling + similar LP [Feldman, Kortsarz, Nutov, SODA '09]
  - Deterministic rounding gives $\tilde{O}\left(n^{4/5+\epsilon}\right)$-approximation
  - We give $\tilde{O}\left(n^{2/3+\epsilon}\right)$-approximation via **randomized rounding**

Conclusion

- $\tilde{O}(\sqrt{n})$-approximation for Directed Spanner
- Small local graphs => better approximation
- Can we do better?
- Hardness: only excludes **polylog(n)**-approximation
- Integrality gap: $\Omega(n^{1/3-\epsilon})$
- Our algorithms are **simple**, can more powerful techniques do better?

Thank you!

- Slides: http://grigory.us