CSCI B609: “Foundations of Data Science”

Lecture 19/20: L_0-sampling, L_1-sparse recovery, Count Sketch

Slides at http://grigory.us/data-science-class.html

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Data Streams

- Stream: $\boldsymbol{m}$ elements from universe $[\boldsymbol{n}] = \{1, 2, \ldots, \boldsymbol{n}\}$, e.g.

$$\langle x_1, x_2, \ldots, x_{\boldsymbol{m}} \rangle = \langle 5, 8, 1, 1, 1, 4, 3, 5, \ldots, 10 \rangle$$

- f_i = frequency of i in the stream = # of occurrences of value i

$$f = \langle f_1, \ldots, f_{\boldsymbol{n}} \rangle$$

Frequency Moments

- Define $F_k = \sum_i f_i^k$ for $k \in \{0,1,2,\ldots\}$
  - $F_0 =$ # number of distinct elements
  - $F_1 =$ # elements
  - F_2 = "Gini index", "surprise index"

ℓ_0-sampling

- Maintain $\widetilde{F_0}$, and (1 ± 0.1)-approximation to F_0.
- Hash items using $h_j : [n] \to \left[0, 2^j - 1\right]$ for $j \in [\log n]$
- For each j, maintain:

$$D_j = (1 \pm 0.1)|\{t | h_j(t) = 0\}|$$

$$S_j = \sum_{t, h_{j(t)} = 0} f_t \times t$$

$$C_j = \sum_{t, h_j(t) = 0} f_t$$

- Lemma: At level $j = 2 + \lceil \log \widetilde{F_0} \rceil$ there is a unique element in the streams that maps to 0 (with constant probability)
- Uniqueness is verified if $D_j = 1 \pm 0.1$. If so, then output S_j / C_j as the index and C_j as the count.

Proof of Lemma

- Let $j = 2 + \lceil \log \widetilde{F_0} \rceil$ and note that $2F_0 < 2^j < 12\, F_0$
- For any i, $\Pr\left[h_j(i) = 0\right] = \frac{1}{2^j}$
- Probability there exists a unique i such that $h_j(i) = 0$,

$$\sum_i \Pr\left[h_j(i) = 0 \; and \; \forall k \neq i, h_j(k) \neq 0\right]$$

$$= \sum_i \Pr\left[h_j(i) = 0\right] \Pr\left[\forall k \neq i, h_j(k) \neq 0 \middle| \, h_j(i) = 0\right]$$

$$\geq \sum_i \Pr\left[h_j(i) = 0\right] \left(1 - \sum_{k \neq i} \Pr\left[h_j(k) = 0 \middle| h_j(i) = 0\right]\right)$$

$$= \sum_i \Pr\left[h_j(i) = 0\right] \left(1 \; - \sum_{k \neq i} \Pr\left[h_j(k) = 0\right]\right) \geq \sum_i \frac{1}{2^j}\left(1 \; - \frac{F_0}{2^j}\right) \geq \frac{1}{24}$$

- Holds even if h_j are only 2-wise independent

Sparse Recovery

- Goal: Find g such that $||f - g||_1$ is minimized among g's with at most k non-zero entries.
- Definition: $Err^k(f) = \min_{g: ||g||_0 \leq k} ||f - g||_1$
- Exercise: $Err^k(f) = \sum_{i \notin S} |f_i|$ where S are indices of k largest f_i

- Using $O(\epsilon^{-1} k \log n)$ space we can find $\tilde{g}$ such that $||\tilde{g}||_0 \leq k$ and
$$||\tilde{g} - f||_1 \leq (1 + \epsilon) Err^k(f)$$

Count-Min Revisited

- Use Count-Min with $d = O(\log n)$, $w = 4k/\epsilon$
- For $i \in [n]$, let $\tilde{f}_i = c_{j,h_j(i)}$ for some row $j \in [d]$
- Let $S = \{i_1, \dots, i_k\}$ be the indices with max. frequencies. Let A_i be the event there doesn't exist t $\in S \setminus i$ with $h_j(i) = h_j(t)$
- Then for $i \in [n]$:

$$\Pr\left[\left|f_i - \widetilde{f}_i\right| \geq \frac{\epsilon Err^k(f)}{k}\right] =$$

$$\Pr[\text{not } A_i] \times \Pr\left[\left|f_i - \widetilde{f}_i\right| \geq \frac{\epsilon Err^k(f)}{k} \middle| not\ A_i\right] +$$

$$\Pr[A_i] \times \Pr\left[\left|f_i - \widetilde{f}_i\right| \geq \frac{\epsilon Err^k(f)}{k} \middle| A_i\right]$$

$$\leq \Pr[\text{not } A_i] + \Pr\left[\left|f_i - \widetilde{f}_i\right| \geq \frac{\epsilon Err^k(f)}{k} \middle| A_i\right] \leq \frac{k}{w} + \frac{1}{4} \leq \frac{1}{2}$$

- Because $d = O(\log n)$ w.h.p. all f_i's approx . up to $\frac{\epsilon Err^k(f)}{k}$

Sparse Recovery Algorithm

- Use Count-Min with $d = O(\log n), w = 4k/\epsilon$
- Let $f' = (\widetilde{f_1}, \widetilde{f_2}, \ldots, \widetilde{f_n})$ be frequency estimates:

$$|f_i - \widetilde{f_i}| \leq \frac{\epsilon Err^k(f)}{k}$$

- Let $\tilde{g}$ be f' with all but the k-th largest entries replaced by 0.
- Lemma: $||\tilde{g} - f||_1 \leq (1 + 3\,\epsilon) Err^k(f)$

$\left\|\tilde{g} - f\right\|_1 \leq (1 + 3\,\epsilon) Err^k(f)$

- Let $S, T \subseteq [n]$ be indices corresponding to k largest values of f and f'.
- For a vector $x \in \mathbb{R}^n$ and $I \subseteq [n]$ denote as x_I the vector formed by zeroing out all entries of x except for those in I.

$$
\begin{aligned}
\left\|f - f'_T\right\|_1 &\leq \left\|f - f_T\right\|_1 + \left\|f_T - f'_T\right\|_1 \\
&= \left\|f\right\|_1 - \left\|f_T\right\|_1 + \left\|f_T - f'_T\right\|_1 \\
&= \left\|f\right\|_1 - \left\|f'_T\right\|_1 + \left(\left\|f'_T\right\|_1 - \left\|f_T\right\|_1\right) + \left\|f_T - f'_T\right\|_1 \\
&\leq \left\|f\right\|_1 - \left\|f'_T\right\|_1 + 2\left\|f_T - f'_T\right\|_1 \\
&\leq \left\|f\right\|_1 - \left\|f'_S\right\|_1 + 2\left\|f_T - f'_T\right\|_1 \\
&\leq \left\|f\right\|_1 - \left\|f_S\right\|_1 + \left(\left\|f_S\right\|_1 - \left\|f'_S\right\|_1\right) + 2\left\|f_T - f'_T\right\|_1 \\
&\leq \left\|f - f_S\right\|_1 + \left\|f_S - f'_S\right\|_1 + 2\left\|f_T - f'_T\right\|_1 \\
&\leq Err^k(f) + k\,\epsilon\,\frac{Err^k(f)}{k} + 2k\,\epsilon\,\frac{Err^k(f)}{k} \\
&\leq (1 + 3\,\epsilon) Err^k(f)
\end{aligned}
$$

Count Sketch [Charikar, Chen, Farach-Colton]

- In addition to $H_i : [n] \to [w]$ use random signs $r_i[n] \to \{-1,1\}$
$$c_{i,j} = \sum_{x : H_i(x) = j} r_i(x) f_x$$
- Estimate:
$$\hat{f}_x = median(r_1(x)c_{1,H_1(x)}, \dots, r_d(x)c_{d,H_d(x)})$$
- Parameters: $d = O\left(\log \frac{1}{\delta}\right), w = \frac{3}{\epsilon^{2}}$
$$\Pr\left[|\widetilde{f_x} - f_x| + \epsilon ||f||_2\right] \geq 1 - \delta$$
- Lemma: $\mathrm{E}[r_i(x)c_{i,H_i(x)}] = f_x$
- Lemma: $\mathrm{Var}[r_i(x)c_{i,H_i(x)}] \leq \frac{F_2}{w}$
- By Chebyshev: $\Pr\left[\left|r_i(x)c_{i,H_i(x)} - f_x\right| \geq \epsilon\sqrt{F_2}\right] \leq 1/3$
- By Chernoff with $d = O\left(\log \frac{1}{\delta}\right)$ error prob. $1 - \delta$.

Count Sketch Analysis

- Fix i and x. Let $X_y = I[H(x) = H(y)]$:

$$r(x)C_{H(x)} = \sum_y r(x)r(y)f_y X_y$$

- Lemma: $\mathrm{E}[r_i(x)c_{i,H_i(x)}] = f_x$

$\mathrm{E}[r(x)C_{H(x)}] = \mathrm{E}[\mathrm{f_x} + \sum_{y \neq x} r(x)r(y)f(y)X_y\ ] = f_x$

- Lemma: $\mathrm{Var}[r_i(x)c_{i,H_i(x)}] \leq \frac{F_2}{w}$

$\mathrm{Var}[r(x)C_{H(x)}] \leq E[(\sum_y r(x)r(y)f_y X_y)^2]$

$= E[\sum_y f_y^2 X_y^2 + \ (\sum_{y \neq z} r(y)r(z)f_y f_z X_y X_z)\ \ ]$
$= F_2/w$